Chiral Observables and Modular Invariants

K.-H. Rehren
Institut für Theoretische Physik, Universität Göttingen (Germany)

Abstract: Various definitions of chiral observables in a given Möbius covariant two-dimensional (2D) theory are shown to be equivalent. Their representation theory in the vacuum Hilbert space of the 2D theory is studied. It shares the general characteristics of modular invariant partition functions, although $\mathrm{SL}(2,\mathbb{Z})$ transformation properties are not assumed. First steps towards classification are made.

1 Introduction

The program of classification of modular invariant partition functions in 2D conformal quantum field theory (see below for more details) has seen steady progress since the original A-D-E classification for SU(2) theories [3]. Apart from explicit classifications for new models [8], classification theorems have been established for the general case [22, 1]. Yet, the feeling persists that the full depth of the problem has not yet been sounded.

It is the intention of the present note to show that general classification theorems of a very similar nature can be derived in a setting which does not refer to modular transformations of Gibbs states at all. Our statements are on the decomposition (described by a "coupling matrix") of the vacuum representation of a conformal 2D quantum field theory upon restriction to its chiral observables. They can be considered with a different perspective as statements on the possible 2D extensions of given left and right chiral algebras. Our mathematical tool is the structure theory of subfactors applied to the inclusion of local algebras of chiral observables into local algebras of 2D observables.

Note that a modular invariant partition function is also described by a coupling matrix which is usually also interpreted as a chiral decomposition of a 2D vacuum representation. But the classification method based on arithmetic properties of the representation matrices S and T of the $\mathrm{SL}(2,\mathbb{Z})$ generators is entirely different and does not rely on this interpretation. In fact, there seem to be exotic (accidental?) modular invariants which do not derive from a 2D theory [1, III].

In contrast to the modular invariants program, we make only rather general structural assumptions on the theory under consideration. We put the emphasis on the local structure [12], rather than the accidental Lie algebra structure of chiral observables. Thus we avoid the, somewhat artificial, restriction to chiral algebras which are related to affine Lie algebras because these are the only ones for which Gibbs functionals $\mathrm{Tr}_\pi\, e^{-\beta L_0}$ ("characters") are known [13]. Likewise, the problem that for most W-algebras it is not clear on which suitable set of "zero mode quantum numbers" for chiral Gibbs functionals the modular group should act, does not pose itself in our approach.

Furthermore, we do not *assume* that the left and right chiral observables are isomorphic, nor that they have isomorphic fusion of their superselection sectors. Instead, we shall *derive* that the maximal (see below) chiral observables automatically possess sectors with identical fusion rules.

To be sure, it is not our intention to depreciate the modular point of view at all. On the contrary, the $\mathrm{SL}(2,\mathbb{Z})$ symmetry between high and low temperature Gibbs states is one of the most fascinating features of chiral models which calls for a sound physical understanding. Indeed, there are general arguments, with reasonable assumptions, in favour of a modular transformation law which generalizes the one for affine Lie algebras [13] as conjectured in [32]. E.g., Cardy [4] argues with transfer matrix methods and invariance under global resummations in lattice models before the continuum limit is taken, and Nahm [23] exploits the operator product algebra of the Schwinger functions to show that Gibbs states transform into Gibbs states. None of these, however, provides a completely satisfactory explanation in terms of the real time local quantum field theory.

On the other hand, the modular group $\mathrm{SL}(2,\mathbb{Z})$ plays a fundamental role even without any Gibbs functionals to act on by a modular transformation of the temperature. Namely, the general theory of superselection sectors collects monodromy data of braid group statistics in numerical matrices S^{stat} and T^{stat}, and as a "maximality" feature of braid group statistics, these matrices represent the modular group [27, 6, 21]. In models where both concepts are defined, one has $S = S^{\mathrm{stat}}$ and $T = T^{\mathrm{stat}}$. E.g., the Kac-Peterson modular matrices [13] for affine Lie algebras can be computed from the statistics of the representations with positive energy of associated local current algebras.

Furthermore, the matrix entries of S^{stat} were found [5, II] to describe the spectrum of the central observables naturally associated with the nontrivial topology of the space S^1. These discoveries are general structure theorems from local quantum field theory and never refer to Gibbs functionals (and hardly to conformal invariance). They show also, however, that a degeneracy (S^{stat} being not invertible) can – and in higher dimensions must – occur which obstructs the existence of an $\mathrm{SL}(2,\mathbb{Z})$ representation. (Algebraic conditions for non-degeneracy are given in [15].)

Thus, even if $\mathrm{SL}(2,\mathbb{Z})$ does not act on chiral characters, it is likely to be around, with various caveats as in the discussion above, as a consequence of fundamentals of local quantum field theory, and an interpretation in terms of Gibbs functionals would be highly desirable. This issue will not be addressed here.

In the classification program for modular invariant 2D partition functions, it is assumed that certain chiral observables $A_L \simeq A_R$ are a priori given along with a collection of representations (sectors) described by their chiral characters (Gibbs functionals for the conformal Hamiltonian L_0 and suitable other quantum numbers such as Cartan charges for current algebras). These characters transform linearly under the group $\mathrm{SL}(2,\mathbb{Z})$ which is essentially generated by the imaginary unit shift (T) and the inversion (S) of the inverse temperature parameter $\beta/2\pi$. One then looks for bilinear combinations of chiral characters with positive integer coefficients $Z_{l,r}$ (the coupling matrix) which are invariant under the simultaneous $\mathrm{SL}(2,\mathbb{Z})$ transformations for both chiral factors (that is, Z commutes with S and T). The resulting modular invariant partition functions are considered as Gibbs functionals for two-dimensional energy and momentum operators in a representation of a 2D conformally invariant quantum field theory. The latter contains the chiral observables along with additional local 2D fields which are nonlocal in each light-cone coordinate separately. In this interpretation, the entries of the coupling matrix Z clearly are the multiplicities of the sectors of the chiral algebras within the representation space of the 2D theory. E.g., one usually imposes the constraint $Z_{0,0} = 1$ on the coupling matrix which ensures this representation to contain a unique vacuum vector.

One of the most important general classification statements [22] asserts that every solution can be turned into a permutation matrix induced by an “automorphism of the fusion rules” with respect to some “suitably extended algebra of chiral observables” $A_L^{\mathrm{ext}} \simeq A_R^{\mathrm{ext}}$. Furthermore, it was found [1] that the non-vanishing diagonal entries of the coupling matrix Z (with respect to the initially given chiral observables) can be characterized in terms of structure data which refer to the chiral extension $A \subset A^{\mathrm{ext}}$ only. In the case of $\mathrm{SU}(2)$, these two statements yield the A-D-E classification of [3].

In this article, we endeavour a somewhat opposite program. We assume a local 2D conformally invariant quantum field theory, denoted by B, to be given in its vacuum representation π^0 on a Hilbert space H. Within this theory we identify chiral observables, denoted by A_L^{max} and A_R^{max}, and show that these are the respective relative commutants of any initially given chiral observables A_R and A_L within the same 2D theory (Corollary 2.7). We then study the superselection sectors of the maximal chiral observables which are contained in H, that is, the branching of the irreducible representation π^0 upon restriction to the subalgebra $A_L^{\mathrm{max}} \otimes A_R^{\mathrm{max}}$. We show that the coupling matrix for the chiral observables A^{max} is described by an isomorphism between the left and right chiral fusion rules (Corollary 3.5), which as a side result implies that A^{max} coincide with A^{ext} in the modular classification statement (Lemma 3.4).

We just use the laws controlling local extensions of local algebras, as established in [17]. The crucial point is the fact that the same coupling matrix which describes the vacuum branching (or the 2D partition function), at the same time describes a distinguished DHR representation of the chiral observables, and an endomorphism of a von Neumann algebra of the form $A_L \otimes A_R$ canonically associated with a subfactor $A_L \otimes A_R \subset B$. The constraints on the coupling matrix arise by the latter endomorphism both being canonical and respecting the tensor product (these notions are explained in Sect. 3).

Unlike locality of the chiral observables, locality of the 2D net is only implicitly exploited and does not yet enter our (outline of the) classification itself. It is well known that left and right chiral sectors (charged fields) cannot be freely composed to yield local 2D fields [28, 25, 22], and a general algebraic condition in terms of a statistics operator was given in [17]. The incorporation of this condition into our present scheme is still awaiting.

As far as these constraints are concerned, very similar arguments also apply to "coset models" in which a tensor product of two commuting subtheories is embedded within a given *chiral* theory. Therefore, the same constraints on the coupling matrix also arise for the branching of the vacuum sector of the ambient theory upon restriction to the pair of subtheories.

The paper is organized as follows. Section 2 sets the physical stage with emphasis on the equivalence of various possible definitions of the chiral observables. In Section 3 the decomposition of the 2D vacuum representation upon restriction to the chiral observables is analyzed in the light of the general theory described in [17]. The central result is a generalization of the "automorphism of the fusion rules" theorem [22]. Section 4 discusses the (first) implications for the classification problem.

The central argument in Section 3 is in fact a theorem on the sector decomposition of the canonical endomorphism of a von Neumann subfactor. This theorem, and the associated notion of a normal canonical tensor product subfactor, is of its own mathematical interest [26] and constitutes the common link between various problems in quantum field theory, such as chiral observables in 2D, and coset models [35] and Jones-Wassermann subfactors [34, 15] in chiral conformal quantum field theory. Its mathematical essence seems to be most appropriately formulated in terms of C* tensor categories. It furthermore reveals a connection to asymptotic subfactors [24] and quantum doubles [15]. This observation may support the expected role of quantum double symmetry in 2D conformal quantum field theory and coset models.

2 Chiral observables

We start with the discussion of various alternatives to define chiral observables within a conformally invariant 2D theory. The reader mainly interested in modular invariants is invited to skip this section, and take its results referred to in Sect. 3 for granted.

We adopt the algebraic approach to quantum field theory in which the local algebras are considered rather than the local (Wightman) fields which possibly generate them. The underlying picture [11] is that the *net of algebras*, i.e., the complete collection of inclusion and intersection relations between algebras associated with smaller and larger space-time regions, is sufficient in principle to reconstruct the full physical content of the theory. Specifications of the model, therefore, have to be formulated as properties of the net of local algebras.

A two-dimensional local conformal quantum field theory is defined on a covering manifold $\widetilde{\mathbb{M}}$ of Minkowski space-time $\mathbb{M} = \mathbb{R}^{(1,1)}$. This manifold is obtained as follows [19, 2]. One first considers Minkowski space-time as the Cartesian product $\mathbb{R} \times \mathbb{R}$ of its two chiral light-cone directions. On each light-cone, the Möbius group $\mathrm{PSL}(2, \mathbb{R})$ acts by the rational transformations $x \mapsto \frac{ax+b}{cx+d}$, thus enforcing the compactification of $\mathbb{R}$ to S^1 by addition of the point $\infty = -\infty$. In the quantum field theory, the chiral Möbius groups are only projectively represented, leading to a covering of S^1 (in which $\mathbb{R}$ will be henceforth identified with the interval $(0, 2\pi)$). The covering manifold $\widetilde{\mathbb{M}}$ is the Cartesian product of the coverings of the two chiral S^1, quotiented by the identification $(x_L, x_R) = (x_L + 2\pi, x_R - 2\pi)$. Each subset $(a, a + 2\pi) \times (b, b + 2\pi)$ represents one copy of Minkowski space-time $\mathbb{M}$ within $\widetilde{\mathbb{M}}$.

The covering manifold $\widetilde{\mathbb{M}}$ possesses a global causal structure such that the causal complement of a double cone $O = (a, b) \times (c, d)$[1] is the double cone $O' = (b, a+2\pi) \times (d - 2\pi, c) \equiv (b - 2\pi, a) \times (d, c + 2\pi)$, and $(O')' = O$.

We may assume that the 2D theory B is given by the isotonous net of local von Neumann algebras $B(O)$ associated with double cones in Minkowski space-time $O = I \times J \subset \mathbb{M}$ where $I \subset \mathbb{R}$ and $J \subset \mathbb{R}$ are open intervals on the respective chiral light-cones. We assume that B is irreducibly represented on a vacuum Hilbert space H, and transforms covariantly under a strongly continuous positive-energy representation U of the 2D conformal group. The latter is the Cartesian product of left and right chiral covering groups $\widetilde{G}_L$, $\widetilde{G}_R$ (with covering projection $p : \tilde{g} \mapsto g$) where $G = \mathrm{PSL}(2, \mathbb{R})$ is the Möbius group. Both chiral Möbius groups G contain a subgroup $\mathrm{U}(1)$ with positive generators L_0, the chiral "conformal Hamiltonians". The corresponding chiral "rotations by 2π" will be denoted for simplicity by $U_L(2\pi)$ and $U_R(2\pi)$. In a local theory, $U_L(2\pi) = U_R(2\pi)$, that is, the diagonal of the kernel of the covering projection p is represented trivially [19].

Conformal covariance means

$$B(g_L I \times g_R J) = \mathrm{Ad}_{U(\tilde{g}_L, \tilde{g}_R)}\, B(I \times J)$$

whenever the elements $\tilde{g} \in \widetilde{G}$ are represented by paths $g_t \in G$ connecting g with the identity which map the respective chiral intervals pointwise into intervals. If, on the other hand, the image of an interval under g_t contains ∞, then the above transformation law is considered as the definition of the algebra on the left hand side, where now $g_L I$ and $g_R J$ are intervals on the covering of the compactified light-cones $S^1 = \mathbb{R} \cup \{\infty\}$. If we denote by $I + 2\pi$ and $J + 2\pi$ the images under chiral rotations by 2π, then it follows that

$$B((I + 2\pi) \times (J - 2\pi)) = B(I \times J),$$

[1] It is always understood that $0 < b - a < 2\pi$ and $0 < d - c < 2\pi$.

that is, the theory B is indeed defined over the conformal covering space $\widetilde{\mathbb{M}}$.

Locality of B on Minkowski space implies that the local algebras also commute whenever the associated double cones in the covering manifold are spacelike separated, i.e., $B(O') \subset B(O)'$. In theories generated by Wightman fields, one even has

Essential duality (duality on the covering space $\widetilde{\mathbb{M}}$):

$$B(O)' = B(O').$$

The same also holds in parity invariant conformal nets [2]. We shall assume essential duality throughout.

Note that any pair O and O' are a left and a right wedge, or likewise the other way round, in a suitable copy of Minkowski space-time in $\widetilde{\mathbb{M}}$, or can be mapped by Möbius transformations into these wedges in any reference copy of Minkowski space-time. Hence, essential duality is equivalent to wedge duality in Minkowski space.

We reserve the term Haag duality, according to its original usage [11], for the stronger property of duality on Minkowski space $\mathbb{M}$ (see below) which is not an automatic feature. It will not be assumed in this paper.

We proceed to define chiral observables.

2.1. Definition: *The (maximal) left chiral observables are*

$$A_L^{\max}(I) := B(I \times J) \cap U(\widetilde{G}_R)'.$$

The (maximal) right chiral observables $A_R^{\max}(J)$ are defined analogously.

First we note that this definition does not depend on the interval J since any two open intervals are connected by a Möbius transformation in $\{e\} \times \widetilde{G}$ which act trivially on $A_L^{\max}(I)$ by definition. Second, the left chiral observables commute with $U_L(2\pi) = U_R(2\pi)$. Consequently, the chiral observables are defined over the compactified light-cone S^1 without covering, and are covariant under the proper Möbius group $G = \mathrm{PSL}(2, \mathbb{R})$. The operators $U_L(2\pi) = U_R(2\pi)$ are multiples of unity in every irreducible subrepresentation of the chiral observables, contained in H. The chiral net of von Neumann algebras $I \mapsto A_L^{\max}(I)$ satisfies chiral locality (commutativity for disjoint intervals) since for given disjoint I_1 and I_2 it is always possible to find intervals J_1 and J_2 such that $O_i = I_i \times J_i$ are space-like to each other, and 2D space-like locality of the net B applies.

Left chiral observables and right chiral observables commute with each other irrespective of their localization since for any I and J there are $\hat{J}$ and $\hat{I}$ such that $I \times \hat{J}$ and $\hat{I} \times J$ are space-like, and again space-like commutativity of B applies.

Clearly, the net $A_L^{\max}$ is Möbius covariant under the representation $U_L \equiv U|_{\widetilde{G}_L}$. By the Reeh-Schlieder theorem, the projections E_L onto the subspaces $\overline{A_L(I)\Omega}$ for any covariant net A_L do not depend on the interval I. By standard arguments, involving the Tomita-Takesaki modular theory [31] and exploiting the geometric action of the modular group associated with conformal double cone algebras [2], one has

2.2. Lemma: *The projection E_L implements a faithful normal conditional expectation $\varepsilon_L : B(I \times J) \to A_L(I)$, that is, for $b \in B(I \times J)$ there is a unique $a =: \varepsilon_L(b) \in A(I)$ such that*

$$E_L b E_L = a E_L.$$

The expectation ε_L preserves the vacuum state, and the vacuum representation of the net A_L is given by

$$\pi_0^L(A_L(I)) = E_L B(I \times J) E_L.$$

The corresponding statements hold for A_R.

Furthermore, for any Möbius covariant chiral net, the local algebras, unless trivial, are known to be type III von Neumann factors, and one has [7, 2]

Essential duality (duality on S^1):

$$\pi_0(A(I))' = \pi_0(A(I'))$$

valid in the vacuum representation π_0 of A.

Hence the chiral observables automatically satisfy essential duality.

2.3. Lemma: *The subspace $\overline{A_L^{\max}(I)\Omega}$ coincides with the subspace of U_R-invariant vectors in H, that is*

$$E'_L = E_L^{\max}$$

where E'_L denotes the projection onto the U_R-invariant subspace. The corresponding statement holds for A_R.

Proof: I owe the following argument to D. Buchholz. We only have to show that every U_R-invariant vector can be approximated in $A_L^{\max}(I)\Omega$. Since $B(O)\Omega$ is dense in H, $E'_L B(O)\Omega$ is dense in $E'_L H$. Consider any vector $\Psi = E'_L b\Omega$ with $b \in B(O)$. Then $\Psi = U_R(g)\Psi = E'_L \alpha_g(b)\Omega$ for all $g \in \widetilde{G}_R$, and $\Psi = E'_L b_T \Omega$ where

$$b_T = \frac{1}{2T} \int_{-T}^{T} dt\, \alpha_{g_t}(b)$$

is an average over the one-parameter group of right chiral dilatations g_t which leave the interval J fixed. Since $\|b_T\| \le \|b\|$, the family b_T has a weak limit point a in the von Neumann algebra $B(O)$ as $T \to \infty$. We are going to show that a is invariant under $\widetilde{G}_R$, hence commutes with U_R and thus belongs to $A_L^{\max}(I)$. It follows that $\Psi = E'_L b_T \Omega = E'_L a\Omega = a\Omega$ is in $A_L^{\max}(I)\Omega$, and the latter space is dense in $E'_L H$.

In order to show the $\widetilde{G}_R$-invariance of a, we first note that

$$\|\alpha_{g_s}(b_T) - b_T\| = \frac{1}{2T} \| \left[\int_{-T+s}^{-T} + \int_{T}^{T+s} \right] dt\, \alpha_{g_t}(b)\| \le \frac{|s|}{T}\|b\|$$

which vanishes as $T \to \infty$. Hence a is dilatation invariant, and

$$\|\alpha_g(a) - a\| = \|\alpha_g \alpha_{g_t}(a) - \alpha_{g_t}(a)\| = \|\alpha_{g_{-t} g g_t}(a) - a\|$$

for all $g \in \widetilde{G}_R$ and all t. For g a translation resp. a special conformal transformation (relative to the dilatations g_t), $g_{-t}gg_t$ tends to the identity as $t \to -\infty$ resp. $t \to +\infty$. For a sufficiently regular to have norm-continuity of α_g (which is the case if b above was regular; such operators still generate a dense subspace of H) it follows that $\|\alpha_g(a) - a\| = 0$, as asserted. □

We want to study the equivalence of the Definition 2.1 with several alternative reasonable definitions. For this purpose, we first compile some useful notions for 2D and for chiral nets.

Generating property: The net B is said to have the generating property if

$$U(\widetilde{G}_L) \subset B(I \times J) \vee B(I' \times J)$$

for any J, and equivalently (taking commutants and using essential duality of B) if

$$B(I \times J) \cap B(I' \times J) \subset U(\widetilde{G}_L)'$$

for any J. (Here I' is either of the two intervals $I^+ = (b, a + 2\pi)$ or $I^- = (b - 2\pi, a)$ if $I = (a, b)$. By the very second formula, the algebra on its left hand side does not depend on this choice, since a suitable left Möbius transformation which maps I onto I^+ and I^- onto I, leaves the intersection of the two algebras invariant.)

Haag duality (duality on Minkowski space $\mathbb{M}$): The net B fulfils Haag duality if

$$B(O)' = B(O^c) \equiv B(O^-) \vee B(O^+)$$

where O^c is the disconnected causal complement of O in Minkowski space with connected components O^-, O^+.

Strong additivity: The net B fulfils strong additivity if

$$B(O_1) \vee B(O_2) = B(O)$$

for O_1 and O_2 the two connected components of the causal complement of an interior point in a double cone O.

Chiral additivity: The net B fulfils chiral additivity if

$$B(I_1 \times J) \vee B(I_2 \times J) = B(I \times J)$$

if I_1, I_2 arise from I by removal of an interior point; and likewise for the two light-cone directions interchanged.

Generating property: A left chiral net A_L of subalgebras of $A_L^{\max}(I)$ has the generating property if

$$U(\widetilde{G}_L) \subset A_L(I) \vee A_L(I');$$

the analogous definition holds for right chiral nets A_R of subalgebras of $A_R^{\max}(J)$.

Haag duality (duality on $\mathbb{R}$): A chiral net A fulfils Haag duality if

$$\pi_0(A(I))' = \pi_0(A(I^c))$$

holds in the vacuum representation. Here I^c denotes the (disconnected) open complement of an interval I in $\mathbb{R}$.

Strong additivity: A chiral net A fulfils strong additivity if

$$A(I_1) \vee A(I_2) = A(I)$$

if I is an interval in S^1 divided into two subintervals I_1, I_2 by removal of an interior point.

It is obvious, that if any net A_L of subalgebras of $A_L^{\max}$ has the generating property, then $A_L^{\max}$ has the generating property and B also has the generating property.

In fact, in view of the previous definition of chiral observables, the generating property for $A^{\max}$ is actually a feature of the 2D net B. In the cyclic subspace of the chiral observables (their vacuum representation), the assumption is always true by essential duality and factoriality. But the generating property for A is required to hold on the full vacuum Hilbert space H of B. It certainly holds if B possesses a conserved stress-energy tensor whose chiral components then are among the chiral observables. It also holds, e.g., in the theory generated by the derivatives of a massless conserved vector current which has nontrivial chiral observables (the derivatives of a U(1) current) but no stress-energy tensor. Namely, in this model, $B(I \times J) = A_L^{\max}(I) \otimes A_R^{\max}(J)$, and $H = H_L \otimes H_R$. Thus H contains only the vacuum representation of the chiral observables. Therefore, we believe that the assumption of the generating property for chiral observables does not exclude any models of serious interest.

The following assertions hardly need to be proven.

2.4. Lemma: (i) *Haag duality is equivalent to strong additivity, both for 2D and chiral conformal nets.*

(ii) *Strong additivity of a 2D net implies chiral additivity.*

Proof: (i) By essential duality, Haag duality is equivalent to $B(O^c) = B(O')$ (in the 2D case). This in turn is strong additivity since, in the covering space $\widetilde{\mathbb{M}}$, the two connected components of O^c touch each other in a point ("space-like infinity"), and thus constitute the causal complement of that point in O'. The same argument applies in the chiral case, as the two connected components of I^c touch each other in S^1 at infinity.
(ii) Let J_1 and J_2 arise by removal of an arbitrary interior point from J, such that $O_1 = I_1 \times J_2$ and $O_2 = I_2 \times J_1$ are the components of the causal complement of an interior point in the double cone $O = I \times J$. Then $B(O_1) \vee B(O_2) \subset B(I_1 \times J) \vee B(I_2 \times J) \subset B(O)$, and strong additivity implies equality. □

In order to compare alternative definitions of chiral observables, we consider the following two chains of inclusions, which hold just by isotony and essential duality:

2.5. Lemma: *With notations as explained below, one has*

$$A_L^{\max}(I_2) \subset \bigcap_J B(I_2 \times J) \subset B_{2,\hat{1}} \cap B_{2,\hat{2}} \subset B_{2,1} \cap B_{2,2} \subset$$

$$\begin{cases} \subset B_{2+3,1} \cap B_{2,2} \equiv B'_{1,2} \cap B_{2,2} \subset A_R(J_2)' \cap B_{2,2} \, . \\ \\ \subset B_{2+3,1} \cap B_{1+2,2} \equiv B'_{1,2} \cap B_{1+2,2} \equiv B_{2+3,1} \cap B'_{3,1} \, . \end{cases}$$

Here we have picked three left chiral intervals $I_1 = (0, a)$, $I_2 = (a, b)$, $I_3 = (b, 2\pi)$ and two right chiral intervals $J_1 = (0, c)$, $J_2 = (c, 2\pi)$ as indicated by the figure, and employ short hand notations $B_{i,j} = B(I_i \times J_j)$, $B_{i,\hat{j}} = B(I_i \times \hat{J}_j)$ with $\hat{J}_j \subset J_j$. The labels $1 + 2$ resp. $2 + 3$ stand for the intervals $(0, b)$ resp. $(a, 2\pi)$.

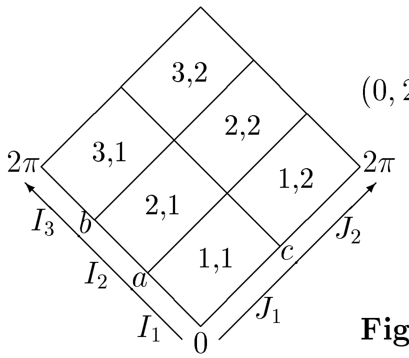

Figure 1: Space-time regions in Lemma 2.5

Of course, the choice of the values $0 < a < b < 2\pi$ and $0 < c < 2\pi$ is completely immaterial since the ensuing partition of one copy of Minkowski space $\mathbb{M}$ within the covering space $\widetilde{\mathbb{M}}$ can be transferred to any other partition of any other copy by left and right Möbius transformations.

A_R in the second line in Lemma 2.5 is any covariant net of subalgebras of $A_R^{\max}(J)$. The consideration of subalgebras of the maximal chiral observables is motivated by our intention to compare with the context of modular invariant partition functions. There one usually starts with some a priori given chiral observables A_R and A_L such as current algebras while the maximal ones might turn out as some "W-algebra" extension thereof. Indeed, we shall later find a condition (Corollary 3.5) when the given chiral observables and the maximal ones coincide.

Of particular interest are the expressions $\bigcap_J B(I_2 \times J)$, $B'_{1,2} \cap B_{1+2,2}$, and $A_R(J_2)' \cap B_{2,2}$ figuring in Lemma 2.5. The first one is possibly nontrivial even in massive 2D theories [30], where it provides a "holographic" sattelite theory (with a conformal symmetry emerging automatically [33]); it has been used as a definition of observables on a horizon in curved space-time [10] in the absence of space-time symmetries. The second one is, up to a Möbius transformation, the relative commutant of a wedge algebra $B(W + a)$ within another wedge algebra $B(W)$ where a is a shift in a light-like direction. The third one is the relative commutant of the opposite chiral observables within a double cone. Each of these would be a sensible definition of chiral observables.

In fact, under suitable conditions, the inclusions above turn into equalities and the various definitions coalesce. (Note that any nontrivial inclusion in the first and second line would require the respective larger algebra not to commute with $U(\widetilde{G}_R)$.)

2.6. Proposition *(referring to the chains of inclusions as in Lemma 2.5):*

(i) *If B has the generating property, then all inclusions in the first line are equalities.*
(ii) *If A_R has the generating property, then all inclusions in the first and second lines are equalities.*
(iii) *If all inclusions in the first and second line are equalities, then B has the generating property.*
(iv) *If B satisfies Haag duality, then $B_{2,\hat{1}} \cap B_{2,\hat{2}} = B'_{1,2} \cap B_{1+2,2}$ where $\hat{J}_1 = (c', c)$, $\hat{J}_2 = (c, c'')$, $0 < c' < c < c'' < 2\pi$; in particular, if in addition the inclusions in the first line are equalities, then all inclusions in the first and third lines are equalities.*
(v) *If all inclusions in the first line are equalities and B satisfies chiral additivity, then $A_L^{\max}$ satisfies Haag duality. The corresponding statement holds for $A_R^{\max}$.*

Versions of assertions (iv) and (v) are also contained in [10].

Proof: (i) The generating property of B implies that $B(I \times J) \cap B(I \times J')$ commutes with $U(\widetilde{G}_R)$ and thus is contained in $A_L^{\max}(I)$.
(ii) $B_{2,2}$ commutes with $A_R(J_1)$, and hence $A_R(J_2)' \cap B_{2,2}$ is contained in $[A_R(J_2)' \cap A_R(J_1)'] \cap B_{2,2}$ which by the generating property of A_R is contained in $U(\widetilde{G}_R)' \cap B_{2,2} = A_L^{\max}(I_2)$.
(iii) We have $U(\widetilde{G}_R) \subset A_L^{\max}(I_2)' = B_{1,2} \vee B_{3,1}$ by definition and by assumption. The claim follows by isotony with $I_1 \times J_2 \subset (b - 2\pi, a) \times J_2$ and $I_3 \times J_1 = (I_3 - 2\pi) \times (J_1 + 2\pi) \subset (b - 2\pi, a) \times J_2$.
(iv) We have $B'_{1,2} \cap B_{1+2,2} = B'_{1,2} \cap B'_{3,1}$. By isotony, this is contained in $B'_{1,2} \cap B'_{3,\check{1}}$ which equals $B_{2,\hat{1}}$ by Haag duality, where $\check{J}_1 = (0, c')$. The same algebra is similarly contained in $B'_{1,\check{2}} \cap B'_{3,1} = B_{2,\hat{2}}$ where $\check{J}_2 = (c'', 2\pi)$. This gives both assertions, by inspection of the chain of inclusions, Lemma 2.5.
(v) Chiral additivity for B is, by passing to the commutants, equivalent to

$$B((a, b) \times J) = B((0, b) \times J) \cap B((a, 2\pi) \times J)$$

for $0 < a < b < 2\pi$ and any interval J. Taking suitable intersections over J to obtain $A_L^{\max}(I)$ by using equality in the first line of the chain of inclusions, yields

$$A_L^{\max}((a, b)) = A_L^{\max}((0, b)) \cap A_L^{\max}((a, 2\pi)).$$

Since the vacuum representation is faithful on $\mathbb{R} \cong (0, 2\pi)$, the same holds in the vacuum representation π_0 (i.e., after multiplication with $E_L^{\max}$). Passing to the commutants in the vacuum representation, and using essential duality for $A_L^{\max}$, one gets strong additivity in the vacuum and hence in any representation. $\square$

We must admit a little lapse in the proof of (v). Namely, the vacuum representation of a chiral net A is known to be faithful on the quasilocal C* algebra on $\mathbb{R} \cong$

$(0, 2\pi)$ which does not contain the von Neumann algebras $A((0, b))$ and $A((a, 2\pi))$. Yet, we are confident that the above conclusion from faithfulness is correct for the intersections. We have tested its validity in the (prototypical) model with a chiral U(1) current j and associated charge $Q = \int j(x)dx$. The operator $\exp itQ$ which is trivially represented in the vacuum representation can be weakly approximated by Weyl operators $\exp itj(f_R)$ as $R \to \infty$ where $f_R(x) = f(x/R)$ and f is a testfunction with $f(x) = 1$ for $|x| < 1$ and $f(x) = 0$ for $|x| > 2$, say. Splitting f_R into two pieces $f_R^\pm$ with supports in $(-2R, a)$ and in $(-a, 2R)$ respectively, yields two Weyl operators $\exp itj(f_R^-)$ and $\exp itj(-f_R^+)$ localized in overlapping left and right halfspaces whose weak limits as $R \to \infty$ should coincide in the vacuum representation, and differ in a charged representation by a factor $\exp itQ$. Nontriviality of these weak limits would invalidate our conclusion in the proof of (v). A calculation, however, shows that, due to scale invariance, the cutoff within the fixed interval $(-a, a)$ in comparison to the increase in R behaves like a cutoff in a scaled interval $(-a/R, a/R)$, and produces an ultraviolet singularity which causes the weak limits of interest to be zero. Since this ultraviolet behaviour is a "universal" effect of scale invariance, we believe that the same mechanism protects the validity of our conclusion also in general models. In any case, (v) will not be needed for the purposes of this paper.

Since we consider the assumption of the generating property for the chiral observables as no serious restriction, we reformulate the statements with this assumption as a default.

2.7. Corollary: (i) *Assume the generating property for some nets $A_L(I)$ and $A_R(J)$ of subalgebras of $B(I \times J)$ which are invariant under the respective opposite Möbius group. Then*

$$A_L^{\max}(I) = \bigcap_J B(I \times J) = A_R(J)' \cap B(I \times J),$$

and similarly for $A_R^{\max}$. In particular, the left and right maximal chiral observables are each other's mutual relative commutants in B.

(ii) *If the net B is Haag dual, then $A_L^{\max}$ and $A_R^{\max}$ are Haag dual, and*

$$A_L^{\max}(I_1) = B(I_2 \times J)' \cap B(I \times J)$$

where I_1, I_2 arise from the interval I by removal of an interior point, and J is an arbitrary interval. The corresponding statement holds for $A_R^{\max}$.

(Again, the assertion of Haag duality for the chiral observables has to be taken with a little caution.)

We conclude this section with a study of the joint position of the subalgebras of left and right chiral observables within $B(O)$. We have

2.8. Proposition: *In the vacuum representation of B, the left and right chiral observables are in a tensor product position, i.e.,*

$$A_L^{\max}(I) \vee A_R^{\max}(J) \simeq A_L^{\max}(I) \otimes A_R^{\max}(J).$$

Proof: The statement follows, by Tomita-Takesaki modular theory [31], from the existence of the conditional expectations ε_L and ε_R, cf. Lemma 2.2. We want to give a less abstract argument.

Since left and right chiral observables mutually commute, it is sufficient to consider products $a_L a_R$ where $a_L \in A_L^{\max}(I)$ and $a_R \in A_R^{\max}(J)$. Since the vacuum state ω is conformally invariant, and since the chiral observables transform under the respective chiral Möbius groups only, we have

$$\omega(a_L a_R) = \omega(\alpha_{g_L \times g_R}(a_L a_R)) = \omega(\alpha_{g_L}(a_L)\alpha_{g_R}(a_R)).$$

For suitable elements g_L and g_R, the localization of the transformed observables tends to space-like infinite separation, hence the cluster property of the vacuum state applies and entails

$$\omega(a_L a_R) = \omega(a_L)\omega(a_R).$$

The factorization of the (normal) vacuum state implies the tensor product factorization of the corresponding algebras. □

3 Representation theory

A subtheory A of a given theory B is described by a net of subalgebras (subfactors) $A(O) \subset B(O)$. Conversely, B may be considered as a (local) extension of a given theory A. In the present paper, A is a net of left and right chiral observables[2] $O \mapsto A(O) = A_L(I) \otimes A_R(J)$, contained in a two-dimensional net $O \mapsto B(O)$.

A general analysis of the representation theory in this situation was initiated in [17]. As a prerequisite it was required that, in generalization of an unbroken global gauge symmetry, there is a consistent family of (normal, faithful) conditional expectations $\varepsilon_O : B(O) \to A(O)$ which commute with space-time symmetries and preserve the vacuum state.

In our situation at hand, these expectations are provided by Takesaki's theorem [31], thanks to the fact that Tomita's modular group for conformal double cone algebras is a subgroup of $\widetilde{G}_L \times \widetilde{G}_R$ and consequently preserves any Möbius covariant subtheory of the form $A_L \otimes A_R$. As in Sect. 2, they are coherently implemented by the projection E_{LR} onto the closure of the subspace $A_L(I)A_R(J)\Omega$ (not depending on $I \times J$), which commutes with Möbius transformations and preserves the vacuum state.

Actually, for the analysis in [17] nets have to be *directed*. We must therefore pass to the 2D and chiral theories on Minkowski space $\mathbb{M}$ and the light-cone axes $\mathbb{R}$, respectively. As is common practice, we denote the quasilocal C* algebra generated by a directed net of von Neumann algebras (say $A(O)$) by the same symbol (say A) as the net itself. We also denote the vacuum representations of A and of B by π_0 and π^0, respectively.

[2]Henceforth, the notation $O = I \times J$ will be understood.

In the algebraic approach to quantum field theory, positive energy representations are conveniently described in terms of DHR endomorphisms [11], provided Haag duality holds. But the restriction of π^0 to the subtheory A is always given by a DHR endomorphism ρ of A

$$\pi^0|_A \simeq \pi_0 \circ \rho$$

even without assuming Haag duality [17]. Moreover, ρ is of the "canonical" form $\rho = \bar{\imath} \circ \imath$. Here $\imath : A \to B$ is the embedding homomorphism and $\bar{\imath} : B \to A$ is a conjugate homomorphism to $\imath$ in the sense [18] that there exist isometric intertwiners $w \in A, w : \mathrm{id}_A \to \bar{\imath} \circ \imath \equiv \rho$ and $v \in B, v : \mathrm{id}_B \to \imath \circ \bar{\imath} \equiv \gamma$ with $w^* v = w^* \gamma(v) = \lambda^{-1} \cdot \mathbb{1}$. The number $\lambda \geq 1$ is the (statistical) dimension of ρ and coincides with the index of the local subfactor $A(O) \subset B(O)$ which is independent of O. (We assume this index, and hence the dimensions of ρ and all its subsectors, to be finite throughout.)

The construction given in [17] starts off from a canonical endomorphism [16] γ_O of the local von Neumann algebra $B(O)$ for any fixed double cone O into its subfactor $A(O)$. γ_O extends to a canonical endomorphism γ of the quasilocal algebra B into A in such a way that on any $B(\hat{O})$, $\hat{O} \supset O$, it yields a canonical endomorphism of $B(\hat{O})$ into $A(\hat{O})$, and consequently the restriction of $\rho = \gamma|_A$ to $A(\hat{O})$ is the corresponding dual canonical endomorphism. It was shown that ρ is a DHR endomorphism localized in the fixed double cone O, and that $w \in A(O)$ and $v \in B(O)$ are local operators.

In the present case, A being a tensor product $A_L \otimes A_R$ of C* algebras, any irreducible representation is also a C* tensor product. As pointed out by R. Longo, there is a theoretical possibility (in case the chiral representations are not "type I", cf. [15]), that the C* tensor products are not spatial. In a large class of models, including current algebras, this possibility can be ruled out [15, Lemma 12], however, and it can presumably never arise when the statistical dimension is finite. Thus we may assume that the corresponding subspaces of H are also tensor products.

Let therefore the irreducible decomposition of the restricted vacuum representation into chiral sectors be given by

$$\pi^0|_{A_L \otimes A_R} \simeq \bigoplus_{l,r} Z_{l,r}\ \pi^L_l \otimes \pi^R_r$$

with a (possibly rectangular) matrix of nonnegative integers $Z_{l,r}$ where l, r run over the irreducible superselection sectors of the left and right chiral observables contained in H. Equivalently, the corresponding DHR endomorphism ρ decomposes as

$$\rho \simeq \bigoplus_{l,r} Z_{l,r}\ \rho^L_l \otimes \rho^R_r$$

with irreducible chiral DHR endomorphisms ρ^L_l and ρ^R_r, and with the same matrix Z. We call Z the **coupling matrix**, and we reserve the labels $l = 0$ and $r = 0$ for the respective vacuum sectors, $\rho^L_0 \simeq \mathrm{id}_{A_L} \equiv \mathrm{id}_L$ and $\rho^R_0 \simeq \mathrm{id}_{A_R} \equiv \mathrm{id}_R$.

Making contact with modular invariants, it should be clear that the coupling matrix also enters the decomposition of the vacuum partition function of a 2D local theory

$$\mathrm{Tr}_{\pi^0}\, e^{-\beta(L_0^L+L_0^R)} = \sum_{l,r} Z_{l,r}\ \mathrm{Tr}_{\pi_l^L}\, e^{-\beta L_0^L}\ \mathrm{Tr}_{\pi_r^R}\, e^{-\beta L_0^R}$$

into chiral characters $\chi_\pi = \mathrm{Tr}_\pi\, e^{-\beta L_0}$ of the representations of the chiral observables.

A similar algebraic situation with a tensor product of two nets of observables embedded into another net also arises in coset models [35] in chiral quantum field theory. These models are given by a net of chiral observables $B(I)$ and a proper subnet $A(I)$ (e.g., the current algebras associated with a compact Lie group G and a subgroup H). The coset theory is defined as the net of relative commutants $C(I) := A(I)' \cap B(I)$. Unless the pair of groups gives rise to a conformal inclusion (in which case $C(I)$ is trivial), the net C possesses a stress-energy tensor of its own which commutes with the stress-energy tensor of A. An argument similar as in Proposition 2.8, making use of the two commuting Möbius groups for A and C, yields the tensor product position of A and C within B. Again, the branching of the vacuum sector of B is described by a coupling matrix, and our results below can be easily adapted to coset models.

We are going to study the branching of the vacuum representation $\pi^0|_A$ in terms of the endomorphism ρ. It turns out convenient to do this in a framework of endomorphisms of von Neumann algebras. For this purpose we use the fact that ρ as a DHR endomorphism of the quasilocal algebra A has the same decomposition into irreducibles as its restriction $\rho_O = \rho|_{A(O)}$ as a (dual canonical) endomorphism of a local von Neumann algebra. This statement is standard if one assumes Haag duality and strong additivity. But it has also been established without these assumptions in the chiral case, making use of conformal symmetry and essential duality instead, provided the statistical dimension is finite [9]. The latter argument carries over without difficulty to the 2D case. We just state this result without repeating its proof.

3.1. Lemma: *Let A be a local net on $\mathbb{M}$ or $\mathbb{R}$. Assume either that A is the restriction of a conformal net on $\widetilde{\mathbb{M}}$ resp. S^1, or that A satisfies Haag duality and strong additivity. Let σ, τ be two DHR endomorphisms (in the conformal case: with finite statistical dimension), localized in some double cone or interval O, and σ_O, τ_O their restrictions to $A(O)$. Then the intertwiner spaces (σ,τ) and (σ_O,τ_O) coincide. In particular, σ and σ_O have the same decomposition into irreducibles.*

Since our nets B and A_L, A_R are conformal, the Lemma applies to all DHR endomorphisms with finite dimension. It follows that the decomposition

$$\rho_O \simeq \bigoplus_{l,r} Z_{l,r}\ \rho_l^L \otimes \rho_r^R$$

of the dual canonical endomorphism for the local subfactor $A_L(I) \otimes A_R(J) \subset B(O)$ is again described by the same coupling matrix Z, where now ρ_l^L and ρ_r^R are local restrictions of chiral DHR endomorphisms.

The crucial additional information here is that ρ and hence the dual canonical endomorphism ρ_O respects the tensor product structure $A(O) = A_L(I) \otimes A_R(J)$ in the sense that its irreducible components are equivalent to tensor products of irreducible endomorphisms of the factor algebras. We call a von Neumann subfactor $A \otimes C \subset B$ with this property a **canonical tensor product subfactor (CTPS)**[3] with associated coupling matrix Z.

The subfactors $A_L(I) \otimes A_R(J) \subset B(O)$, or $A(I) \otimes C(I) \subset B(I)$ for coset models, are examples of CTPS's. Other examples in conformal quantum field theory are Jones-Wassermann subfactors arising from partitions of S^1 into four intervals [34, 15].

Since we assume the index to be finite, only finitely many sectors can contribute which all must have finite dimension, hence the coupling matrix is a finite matrix. Since we have assumed the defining representation of B to contain a unique vacuum vector, it follows that its restriction to the chiral observables contains the joint vacuum representation exactly once, hence $Z_{0,0} = 1$. This implies that the multiplicity of $\mathrm{id}_L \otimes \mathrm{id}_R$ in ρ is one, hence the embedding $A_L \otimes A_R \subset B$ is irreducible (both for the local von Neumann algebras and for the quasilocal C* algebras).

We summarize the discussion so far:

3.2. Proposition: *The local subfactors $A_L(I) \otimes A_R(J) \subset B(O)$ are irreducible canonical tensor product subfactors. The irreducible sector decomposition of their dual canonical endomorphisms is described by the same finite coupling matrix Z as the decomposition of the restricted vacuum representation $\pi^0|_{A_L \otimes A_R}$ of B.*

We are going to study the constraints on Z being the coupling matrix of a canonical TPS. These constraints are then read back as constraints on the representation $\pi^0|_{A_L \otimes A_R}$ or on the 2D partition function.

In the sequel when we write $A_L \otimes A_R \subset B$, we have in mind the local subfactor $A_L(I) \otimes A_R(J) \subset B(O)$, or with suitable modifications $A(I) \otimes C(I) \subset B(I)$ in coset models. But we are actually going to establish general statements on coupling matrices of CTPS's without reference to quantum field theory.

We shall several times need "Frobenius reciprocity", cf. [18], which we recall in

3.3. Lemma: *Let A, B, C be unital C^* or von Neumann algebras and $\alpha : A \to B$, $\beta : B \to C$, $\gamma : A \to C$ unital homomorphisms. Denote by $\langle \gamma, \alpha\beta \rangle$ the dimension of the linear space of intertwiners $t \in C$, $t : \gamma \to \alpha\beta$. Then*

$$\langle \bar{\alpha}\gamma, \beta \rangle = \langle \gamma, \alpha\beta \rangle = \langle \gamma\bar{\beta}, \alpha \rangle$$

provided the conjugate homomorphisms $\bar{\alpha} : B \to A$ or $\bar{\beta} : C \to B$ exist.

Here, as before, conjugates are defined in terms of a pair of intertwiners [18], say $w : \mathrm{id}_A \to \bar{\alpha}\alpha$, $v : \mathrm{id}_B \to \alpha\bar{\alpha}$ which satisfy the relations $\alpha(w)^* v = \mathbb{1}_B$, $\bar{\alpha}(v)^* w = \mathbb{1}_A$.

[3] An elementary example of a subfactor $A \otimes C \subset B$ which is *not* canonical in this sense was suggested to me by H.J. Borchers: take $C = A$, and B the crossed product of $A \otimes A$ by the flip automorphism. Then the dual canonical endomorphism is the direct sum of the identity and the flip. The latter does not respect the tensor product.

For $X \subset B$ the relative commutant $X' \cap B$ is commonly denoted by X^c. We have

3.4. Lemma: *Let $A_L \otimes A_R \subset B$ be a CTPS with finite index, and $Z_{l,r}$ its coupling matrix. Then, $Z_{0,r} \neq 0$ implies $r = 0$ if and only if $\mathbb{1} \otimes A_R = (A_L \otimes \mathbb{1})^c$. The corresponding statement holds exchanging A_L and A_R.*

Proof: We have to show that $Z_{0,r} \neq 0$ for some $r \neq 0$ (that is, $\rho_r^R \not\simeq \mathrm{id}_R$) if and only if the inclusion $\mathbb{1} \otimes A_R \subset (A_L \otimes \mathbb{1})^c$ is proper. Note that equality holds if and only if $X := (A_L \otimes \mathbb{1}) \vee (A_L \otimes \mathbb{1})^c$ equals $A_L \otimes A_R$ (since A_L is a factor).

Consider now the intermediate subfactor $A_L \otimes A_R \subset X \subset B$. In terms of the inclusion maps $\imath_1 : A_L \otimes A_R \to X$ and $\imath_2 : X \to B$, we have

$$\rho_1 \equiv \bar{\imath}_1 \imath_1 \prec \bar{\imath}_1 \bar{\imath}_2 \imath_2 \imath_1 = \bar{\imath} \imath \equiv \rho.$$

If $A_L \otimes A_R \subset X$ is proper, then $\imath_1$ is nontrivial and ρ_1 contains a nontrivial subsector $\mathrm{id}_L \otimes \rho_r^R$ which is also a subsector of ρ giving rise to a nonvanishing matrix entry $Z_{0,r}$. Conversely, if $Z_{0,r} \neq 0$, then $\mathrm{id}_L \otimes \rho_r^R \prec \rho$. By Frobenius reciprocity (Lemma 3.3), $\imath \prec \imath \circ (\mathrm{id}_L \otimes \rho_r^R)$, hence there is a nonvanishing intertwiner $\psi \in B$ which satisfies

$$\psi(a_L \otimes a_R) = (a_L \otimes \rho_r^R(a_R))\psi.$$

Putting $a_R = \mathbb{1}$, this implies that $\psi \in (A_L \otimes \mathbb{1})^c$, thus $\psi \in X$, and hence $\imath_1 \prec \imath_1 \circ (\mathrm{id}_L \otimes \rho_r^R)$. Again invoking Frobenius reciprocity, $\mathrm{id}_L \otimes \rho_r^R \prec \rho_1$. Thus $A_L \otimes A_R \subset X$ is proper. $\square$

The Lemma allows us to characterize the maximal chiral observables by a normality property of the local subfactors, see Corollary 3.5 below. We recall that an inclusion $A \subset B$ is called normal if $(A^c)^c = A$. In general, $A^{cc} \supset A$. It follows that $(A^{cc})^c \subset A^c$ and $(A^c)^{cc} \supset A^c$, hence $A^{ccc} = A^c$ which is obviously equivalent to the statement that a relative commutant is always normal.

We call (with a slight abuse of terminology) a tensor product subfactor $A \otimes C \subset B$ **normal** if $A \otimes \mathbb{1}$ and $\mathbb{1} \otimes C$ are each other's relative commutants in B.

Hence, the local subfactors of chiral observables within 2D conformal quantum field theories, $A_L^{\max}(I) \otimes A_R^{\max}(J) \subset B(O)$ are examples of normal and canonical TPS's. Also coset models give rise to local subfactors which are normal CTPS's. Namely, one obtains normality by extending (if necessary) $A(I)$ by the relative commutant of $C(I)$.

Normality of the local subfactors is characteristic for the maximal chiral observables, and a criterium in terms of the coupling matrix is given in

3.5. Corollary: *The following are equivalent.*

(i) $A_L = A_L^{\max}$ *and* $A_R = A_R^{\max}$.
(ii) *The local subfactors $A_L(I) \otimes A_R(J) \subset B(O)$ are normal CTPS's.*
(iii) *The coupling matrix satisfies $Z_{0,r} = \delta_{0,r}$ and $Z_{l,0} = \delta_{l,0}$.*
(iv) *The coupling matrix describes an isomorphism of the left and right chiral fusion rules (in the sense of Theorem 3.6 below).*

Proof: (i) and (ii) are equivalent by Corollary 2.7. (ii) and (iii) are equivalent by Lemma 3.4. (iii) and (iv) are equivalent by the following Theorem. □

(The equivalence (i) $\Leftrightarrow$ (iii) could have been argued already from Lemma 2.3.)

3.6. Theorem: *Let $A_L \otimes A_R \subset B$ be a CTPS with finite index, and $Z_{l,r}$ its coupling matrix, that is*

$$\rho = \bar\imath \circ \imath \simeq \bigoplus_{l,r} Z_{l,r}\ \rho_l^L \otimes \rho_r^R$$

where $\imath : A_L \otimes A_R \to B$ denote the inclusion map and $\bar\imath$ its conjugate. If the coupling matrix satisfies

$$Z_{0,r} = \delta_{0,r} \quad \text{and} \quad Z_{l,0} = \delta_{l,0}$$

(that is, the CTPS is normal and irreducible), then
(1) Z is a permutation matrix. It induces a bijection $\hat{\cdot}$ with inverse $\check{\cdot}$ between the systems of sectors $\{\rho_l^L\}$ and $\{\rho_r^R\}$ contributing to the decomposition of ρ such that

$$Z_{l,r} = \delta_{\hat l,r} = \delta_{l,\check r}.$$

(2) Both systems of sectors $\{\rho_l^L\}$ and $\{\rho_r^R\}$ are closed under conjugation and under decomposition of products (fusion). They satisfy the same fusion rules

$$\rho_l^L \rho_k^L \simeq \bigoplus_m N_{lk}^m\ \rho_m^L \qquad \text{and} \qquad \rho_r^R \rho_s^R \simeq \bigoplus_t \hat N_{rs}^t\ \rho_t^R$$

with $N_{lk}^m = \hat N_{\hat l \hat k}^{\hat m}$. In particular, the bijection $\hat{\cdot}$ respects conjugation, and the dimensions of the corresponding sectors coincide:

$$d(\rho_{\hat l}^R) = d(\rho_l^L).$$

(3) The homomorphisms $\imath_l^L := \imath\circ(\rho_l^L \otimes \mathrm{id}_R) : A_L \otimes A_R \to B$ are irreducible and mutually inequivalent. The same holds for $\imath_r^R := \imath\circ(\mathrm{id}_L \otimes \rho_r^R)$, and $\imath_r^R \simeq \imath_{\check r}^L$. Moreover,

$$\imath\circ(\rho_l^L \otimes \rho_r^R) \simeq \bigoplus_k N_{\check r l}^k\ \imath_k^L \simeq \bigoplus_s \hat N_{\hat l r}^s\ \imath_s^R.$$

Proof: The proof adopts and extends methods taken from [20].

Let the index sets $\{l\}$ and $\{r\}$ label the irreducible sectors ρ_l^L of A_L and ρ_r^R of A_R, respectively, obtained by closure under reduction of products of those sectors which contribute to ρ. If among these there are any "new" sectors not already contributing to ρ, we extend the coupling matrix Z by zero columns and rows, but we are eventually going to show that there are no such new sectors.

Only finitely many columns and rows of Z are non-zero. Since $\rho = \bar\imath\circ\imath$ is self-conjugate, along with $\rho_l^L \otimes \rho_r^R$ also its conjugate must contribute with the same multiplicity, and hence $Z_{l,r} = Z_{\bar l,\bar r}$. In particular, both systems $\{\rho_l^L\}$ and $\{\rho_r^R\}$ are closed under conjugation.

Let the homomorphisms $\imath^L_l : A_L \otimes A_R \to B$ be as in (3). We compute

$$\langle \imath^L_l, \imath^L_{l'} \rangle = \langle \imath\circ(\rho^L_l \otimes \mathrm{id}_R), \imath\circ(\rho^L_{l'} \otimes \mathrm{id}_R)\rangle = \langle \rho^L_l \otimes \mathrm{id}_R, \bar\imath\circ\imath\circ(\rho^L_{l'} \otimes \mathrm{id}_R)\rangle =$$
$$= \sum_{k,s} Z_{k,s} \langle \rho^L_l \otimes \mathrm{id}_R, \rho^L_k \rho^L_{l'} \otimes \rho^R_s \rangle = \sum_{k,s} Z_{k,s} \langle \rho^L_l, \rho^L_k \rho^L_{l'} \rangle \langle \mathrm{id}_R, \rho^R_s \rangle.$$

To this sum contributes only $s = 0$ since $\langle \mathrm{id}_R, \rho^R_s \rangle = \delta_{s,0}$, and by the assumed properties of Z also $k = 0$ is the only contribution. Hence

$$\langle \imath^L_l, \imath^L_{l'} \rangle = \langle \rho^L_l, \rho^L_{l'} \rangle = \delta_{l,l'}.$$

Thus the homomorphisms $\imath^L_l$ are irreducible and mutually inequivalent. The symmetric argument applies to $\imath^R_r$. Next we compute

$$\langle \imath^L_l, \imath^R_{\bar r} \rangle = \langle \rho^L_l \otimes \mathrm{id}_R, \bar\imath\circ\imath\circ(\mathrm{id}_L \otimes \bar\rho^R_r)\rangle = \sum_{k,s} Z_{k,s} \langle \rho^L_l, \rho^L_k \rangle \langle \mathrm{id}_R, \rho^R_s \bar\rho^R_r \rangle = Z_{l,r}\ .$$

As we have seen that both sets of homomorphisms $\{\imath^L_l\}$ and $\{\imath^R_r\}$ consist of mutually inequivalent irreducibles, each $\imath^L_l$ can be equivalent to at most one $\imath^R_{\bar r}$. Hence for fixed index l, at most one entry $Z_{l,r}$ can be different from zero and must be one. It follows also that no $\imath^L_l$ associated with a "new" sector ρ^L_l can be equivalent to any of the $\imath^R_r$, old or new, and vice versa.

For the "old" sectors, we write

$$r = \hat l \quad \text{and} \quad l = \check r \qquad \text{iff} \quad Z_{l,r} = 1, \quad \text{that is, iff} \quad \imath^L_l \simeq \imath^R_{\bar r}.$$

That this assignment between old sectors is bijective follows from transitivity of equivalence of sectors. Since we have already seen that Z is conjugation invariant, this assignment respects conjugation, that is

$$\bar\rho^R_{\hat l} = \rho^R_{\bar{\hat l}} = \rho^R_{\hat{\bar l}} \qquad \text{etc.}$$

Next, we consider homomorphisms $\imath_{l,r} := \imath\circ(\rho^L_l \otimes \rho^R_r) : A_L \otimes A_R \to B$ and compute

$$\imath_{l,r} = \imath^R_r\circ(\rho^L_l \otimes \mathrm{id}_R) \simeq \imath^L_{\check{\bar r}}\circ(\rho^L_l \otimes \mathrm{id}_R) = \imath\circ(\bar\rho^L_{\check r} \rho^L_l \otimes \mathrm{id}_R) \simeq \bigoplus_k N^k_{\check{\bar r} l}\ \imath^L_k.$$

The symmetric argument produces also the decomposition

$$\imath_{l,r} = \imath^L_l\circ(\mathrm{id}_L \otimes \rho^R_r) \simeq \imath\circ(\mathrm{id}_L \otimes \bar\rho^R_{\hat l} \rho^R_r) \simeq \bigoplus_s \hat N^s_{\hat{\bar l} r}\ \imath^R_s.$$

In the first of these two decomposition formulae of the same object, no "new" label k can appear, since we have seen that such a term $\imath^L_k$ is not equivalent to any term $\imath^R_s$ in the second decomposition formula, and vice versa. This shows that the sets of sectors contributing to the coupling matrix are already closed under reduction of products.

Furthermore, comparison of the two decomposition formulae shows equality of the multiplicities $N^k_{\bar{r}l}$ and $\hat{N}^s_{\hat{l}r} \equiv \hat{N}^{\bar{s}}_{\bar{r}\hat{l}}$ if $\bar{s} = \hat{k}$. Hence the bijection $\hat{\cdot}$ between the sectors induces an isomorphism of the fusion rules.

Since finally the fusion rules of a finite system determine the dimensions uniquely, also the equality of the dimensions follows. □

We have thus reproduced a result found previously in the classification program for modular invariant partition functions with heavy use of $\mathrm{SL}(2,\mathbb{Z})$ machinery [22], reducing every modular invariant to an "automorphism of the fusion rules" for suitably extended chiral observables. Our analysis is, however, much stronger since its assumptions are much weaker. Furthermore, it implies that the "suitably extended" chiral observables are indeed the maximal chiral observables defined in 2.1, and coincide with the relative commutants of the initially given chiral observables (Corollary 2.7(i)).

Second, if possibly the maximal left and right chiral observables are not isomorphic, then the result still implies an isomorphism of the respective fusion rules. The corresponding statement is even more interesting in the case of coset models where typically $A \subset B$ is a theory with well-known fusion rules, while the coset theory $C = A^c$ is in general a W-algebra whose superselection structure is a priori unknown. The theorem establishes that the fusion rules of this W-algebra are isomorphic to those of a local extension of the given theory A, namely the relative commutant A^{cc} of C, which is in turn controllable in terms of the representations of A itself. For coset models based on current algebras, our result seems to be the algebraic backbone of the modular reasoning as in [29].

Finally, we emphasize that the sectors in Theorem 3.6 were never referred to as being restrictions of DHR sectors. Neither was it required that their fusion be abelian. The theorem is thus of a quite more general nature than its specific application to conformal quantum field theory as treated in this paper.

4 Towards classification

Modular invariant partition functions associated with affine Lie algebras ($A_L \simeq A \simeq A_R$), as far as they have been classified, exhibit a classification scheme which refers to certain graphs and their exponents (eigenvalues of the square of the adjacency matrix) [3, 8]. An essential statement is on the non-vanishing diagonal entries of the coupling matrix Z.

A rather general formulation can be found in [1, II]. It entails that $Z_{\lambda,\lambda} \neq 0$[4] if and only if the DHR sector λ of A belongs to a set of "exponents" associated with the chiral extensions $A \subset A^{\mathrm{ext}}$. The set of exponents is a subset of the sectors of A.

[4] In affine models the DHR sectors of the initially given chiral observables are given in terms of weights λ of semisimple Lie algebras. Throughout this section, we adopt the labels λ for DHR sectors in order to make the present generalizations more transparent.

By modular invariance, the sectors of A label at the same time also the irreducible representations of their own fusion algebra, the modular matrix S playing the role of a "generalized Fourier transformation" between the fusion algebra itself and its dual. On the other hand, modular invariance of the partition function implies that the coupling matrix coincides with its Fourier transform (up to a conjugation). Hence, the above statement on the sector λ being an exponent can as well be interpreted as a statement on the irreducible representation λ of the fusion algebra and on non-vanishing entries of the Fourier transformed coupling matrix. In the following, we set out to formulate a generalization of this version of the statement to the more general situation we discussed in this paper (without parity symmetry between left and right chiral algebras, and without assumption of modular invariance).

Let $A_L \otimes A_R \subset A_L^{\max} \otimes A_R^{\max} \subset B$ denote some initially given chiral observables embedded into a two-dimensional local theory B (satisfying the assumptions of section 2) along with their maximal chiral extensions obtained by passing to the relative commutants in B.

Let W_L and W_R denote the fusion algebras of all irreducible DHR sectors λ_L, λ_R of the initially given chiral observables (or fusion subalgebras containing all sectors which contribute to the coupling matrix Z). Let $W_L^{\max}$ and $W_R^{\max}$ denote the fusion algebras of the irreducible sectors τ_L, τ_R of the extended (= maximal) chiral observables which contribute to the coupling matrix (i.e., which are contained in the vacuum representation of B). According to Theorem 3.6 and Corollary 3.5, the fusion algebras $W_L^{\max}$ and $W_R^{\max}$ are isomorphic under the bijection $\hat{\cdot}$. We use this bijection to identify $W_L^{\max}$ with $W_R^{\max}$, so the coupling matrix with respect to $A^{\max}$ becomes the unit matrix $\mathbb{1}$.

To be on safe grounds, we assume that W_L and W_R contain only finitely many sectors, and that these have finite dimensions. This implies the same for $W^{\max}$, and ensures that all extensions have finite index.

Restriction and extension prescriptions between DHR sectors of a theory B and a subtheory A were given in [17], and further analyzed in [1]. We are going to apply this theory to the chiral extensions $A_L^{\max}$ of A_L, and $A_R^{\max}$ of A_R.

The restriction is just the restriction of representations and coincides with the "canonical" prescription in terms of the inclusion homomorphism $\imath$ and its conjugate, given by $\tau \mapsto \sigma_\tau = \bar{\imath}\circ\tau\circ\imath$. It was named σ-restriction in [1]. In the present situation, σ-restriction maps $W^{\max}$ into W.[5]

In contrast, the extension prescription $\lambda \mapsto \alpha_\lambda$ [17] differs from the canonical induction $\lambda \mapsto \imath\circ\lambda\circ\bar{\imath}$; it was named α-induction in [1] for distinction. In particular, unlike canonical induction, α-induction respects sector composition, and the trivial sector of the subtheory extends to the trivial sector of the extended theory. Furthermore, α-extensions of DHR sectors of the subtheory in general are not DHR but only

[5]Here and in the sequel, we often suppress the subscripts L and R when both chiralities are understood.

half-space localized (solitonic) sectors, due to a monodromy obstruction [17]. Let V_L and V_R denote the, possibly non-abelian, fusion algebras of all sectors (labelled β) generated by reduction of products of α-extended DHR sectors from W_L and W_R.

In [1], a reciprocity formula for α-induction and σ-restriction was found:

$$\langle \alpha_\lambda, \tau \rangle = \langle \lambda, \sigma_\tau \rangle$$

provided λ and τ are DHR sectors of the respective theories. It entails that α_λ and $\imath \circ \lambda \circ \bar{\imath}$, while otherwise different, contain the same DHR subsectors. It also entails that, in the present setting, the fusion algebras V contain the abelian subalgebras $W^{\max}$.

Let B_L and B_R denote the rectangular "branching matrices", describing chiral σ-restriction, with non-negative integer entries $\langle \lambda, \sigma_\tau \rangle$ which connect the irreducible DHR sectors $\tau \in W^{\max}$ with $\lambda \in W$. Then the (in general rectangular, $\dim W_L \times \dim W_R$) coupling matrix with respect to the initially given chiral observables is

$$Z = B_L B_R^t,$$

that is, $Z_{\lambda_L,\lambda_R} \neq 0$ if and only if the sectors λ_L and λ_R arise by restriction from a pair of sectors of the maximal chiral observables which are identified by the bijection $\hat{}$ of Theorem 3.6. This is just the "block form" of the coupling matrix expected by restricting first π_B^0 to the maximal chiral observables, and subsequently restricting the sectors so obtained to the initially given chiral observables.

Each fusion algebra has a "regular representation" defined by representing a sector by its matrix of fusion multiplicities with the other sectors. W and $W^{\max}$ being abelian, all their irreducible representations are one-dimensional and contribute with multiplicity one to the regular representations. The values of the generators of the fusion algebra in the irreducible representations provide "character tables" which are non-degenerate square matrices. We denote the one-dimensional representations of W by $\phi \in \widehat{W}$, and their character tables by X.

The character table defines a "generalized Fourier transform" between any abelian fusion algebra and its representations. The Fourier transformed coupling matrix is thus defined as

$$\widehat{Z} = (X_L B_L)(X_R B_R)^t.$$

Its matrix entries are the values of the restriction of the vacuum sector of the 2D theory B, as a DHR sector of $A_L \otimes A_R$, in the irreducible representations $\phi_L \otimes \phi_R$ of the tensor product $W_L \otimes W_R$ of the chiral DHR fusion algebras. A priori, the entries need not to be integers.

Let $\bar{\phi} \in \widehat{W}$ denote the conjugate representation of ϕ. Since the adjoint in the fusion algebra is given by sector conjugation, we have $\phi(\bar{\lambda}) = \overline{\phi(\lambda)} = \bar{\phi}(\lambda)$. This means $\widehat{C}X = XC = \overline{X}$ where C and $\widehat{C}$ are the conjugation matrices for the DHR sectors of the initially given chiral observables A and for the representations of their fusion algebras W, respectively. Furthermore, restriction respects sector conjugation, hence $BC^{\max} = CB$ where $C^{\max}$ is the conjugation matrix for the sectors $\tau \in W^{\max}$ of the maximal chiral observables $A^{\max}$.

Thus, since the branching matrices are real, we arrive at

$$\widehat{Z}\widehat{C} = (X_L B_L)(X_R B_R)^+ \qquad \text{or equivalently} \qquad \widehat{Z} = (X_L B_L) C (X_R B_R)^+.$$

It follows that a matrix entry of $\widehat{Z}\widehat{C}$ to be non-zero requires that the corresponding complex row vectors of $X_L B_L$ and $X_R B_R$ are not orthogonal, and a fortiori non-zero. If both the chiral branching and the chiral fusion algebras are isomorphic, e.g., if the theory B is parity symmetric, then a diagonal matrix entry of $\widehat{Z}\widehat{C}$ vanishes if *and only if* the corresponding row vector of XB vanishes.

A modular (transformation) matrix S, if it exists, establishes a natural identification between the generators of a fusion algebra and its representations, and $X = S$. Since $S^2 = C = \widehat{C}$, modular S-invariance is the statement that the coupling matrix $Z = SZS^* = \widehat{Z}\widehat{C}$ equals its own Fourier transform up to a conjugation.

This remark implies that the Proposition 4.1 below reduces to the above-mentioned statement on "exponents" in [1] in the case with modular invariance.

We have first to adapt definitions made in [1, II] to our more general setting. We introduce certain subsets of $\widehat{W}$ which reflect the structure of the chiral extensions.

For a given irreducible DHR sector $\tau \in W^{\max}$, we define the **σ-supports** $\mathrm{Supp}_L(\tau)$ and $\mathrm{Supp}_R(\tau)$ as the subsets of those irreducible representations of W_L and W_R which do not vanish on the respective restrictions σ_τ of τ to the initially given left and right chiral observables, that is, those rows of XB which have non-zero entry in column τ. The notion "support" is motivated by considering the abelian fusion algebra W as an algebra of functions on the set $\widehat{W}$ of its one-dimensional representations. Thus $\mathrm{Supp}(\tau) \subset \widehat{W}$ is indeed the support of the function $\sigma_\tau \in W$. (The σ-supports were called $\mathrm{Eig}(\tau)$ in [1].)

As shown in [1], α-induction of sectors induces a homomorphism of fusion algebras $W \to V$. Composing this homomorphism with the regular representation of V yields another representation, π_α, of W. We define the **α-spectra** Spec_L and Spec_R as the subsets of those irreducible representations of W_L and W_R which are contained in the α-induced representations π^L_α and π^R_α. (The α-spectra were called Exp in [1] and are the "exponents" mentioned above.)

Now, by virtue of α-σ-reciprocity [1], we are going to derive

4.1. Proposition: (i) *A matrix entry of $\widehat{Z}\widehat{C}$ vanishes unless for some sector $\tau \in W^{\max}$, both matrix indices belong to the respective left and right σ-supports* $\mathrm{Supp}(\tau)$. *It also vanishes unless both matrix indices belong to the left and right α-spectra* Spec.

(ii) *If (fusion and branching of) the left and right chiral theories are isomorphic, then a diagonal matrix entry of $\widehat{Z}\widehat{C}$ is non-zero if and only if the corresponding representation of W belongs to the union* $\bigcup_\tau \mathrm{Supp}(\tau)$.

In fact, there are many interesting cases when $\bigcup_\tau \mathrm{Supp}(\tau) = \mathrm{Spec}$ (some of them being given below), so the last statement can be phrased in terms of the α-spectrum Spec.

The Proposition is the desired generalization of the classification statement [3, 8, 1] for modular invariant partition functions. (The second statement seems not to be sensible with differing left and right chiral fusion and branching matrices, since the product of two different row vectors can clearly vanish without these vectors being zero.)

The Proposition makes assertions about the coupling matrix for the initially given chiral observables $A_L \otimes A_R$ embedded into the 2D theory B, in terms of the chiral extensions $A \subset A^{\max}$ to which α-induction and σ-restriction pertain. Thus the 2D problem is reduced to a chiral problem. An open issue remains, however, a model-independent classification of possible α-spectra, and hence of 2D chiral extensions. The available classifications for affine Lie and Virasoro algebras ("diagonal or automorphism, orbifold, exceptional" [3, 8, 1]) refer to the chiral extensions being in turn trivial, fixpoints under an abelian group, or conformal embeddings, and are expected to be too coarse in the general case.

Proof of the Proposition: (i) The first statement is obvious since by the representation $\widehat{Z}\widehat{C} = (X_L B_L)(X_R B_R)^+$, every matrix entry is the inner product of row vectors whose components are the values of the functions σ_τ, $\tau \in W^{\max}$, evaluated on the respective left and right one-dimensional representations. The inner product vanishes whenever these representations do not belong to the respective σ-supports. The second statement is a consequence of the first in view of the Lemma 4.2 below.
(ii) For isomorphic left and right chiral fusion and branching, $X_L B_L = X_R B_R$, diagonal matrix entries of $\widehat{Z}\widehat{C}$ are norm squares of row vectors of XB which vanish if *and only if* all their entries vanish, hence if *and only if* the corresponding representation of W does not belong to any of the σ-supports $\mathrm{Supp}(\tau)$, $\tau \in W^{\max}$. $\square$

We have used

4.2. Lemma: $\bigcup_\tau \mathrm{Supp}(\tau) \subset \mathrm{Spec}$.

Proof: The one-dimensional representations ϕ of an abelian fusion algebra with generators λ, considered as vectors with entries $\phi(\lambda)$, are pairwise orthogonal [14]. This property enables us to decide whether a representation ϕ is contained in the α-induced representation $\pi_\alpha(\lambda)$ with matrix entries $\langle \alpha_\lambda \beta_1, \beta_2 \rangle$, by contracting the matrix-valued vector $(\pi_\alpha(\lambda))_\lambda$ with the vector $(\overline{\phi(\lambda)})_\lambda$. Thus ϕ belongs to the α-spectrum Spec if and only if the resulting matrix

$$(\overline{\phi} \cdot \pi_\alpha)_{\beta_1\beta_2} \equiv \sum_\lambda \overline{\phi(\lambda)} \pi_\alpha(\lambda)_{\beta_1\beta_2} = \sum_\lambda \overline{\phi(\lambda)} \langle \alpha_\lambda \beta_1, \beta_2 \rangle$$

is non-zero. But for $\beta_1 = \mathrm{id}_{A^{\max}}$, and $\beta_2 = \tau$ an irreducible sector from $W^{\max} \subset V$, the matrix entry of the α-induced representation equals $\langle \lambda, \sigma_\tau \rangle$ by α-σ-reciprocity, and the contracted matrix entry equals $\overline{\phi(\sigma_\tau)}$. Hence, if ϕ belongs to any of the σ-supports $\mathrm{Supp}(\tau)$, then ϕ belongs to the α-spectrum Spec. $\square$

We list here two "extremal", but by no means exhaustive, conditions to ensure equality in Lemma 4.2, that is, $\bigcup_\tau \mathrm{Supp}(\tau) = \mathrm{Spec}$:

4.3. Lemma: *If α-induction is surjective (considered as a linear map from W into V), then* $\mathrm{Supp}(\mathrm{id}_{A^{\max}}) = \bigcup_\tau \mathrm{Supp}(\tau) = \mathrm{Spec}$.

If σ-restriction is surjective (considered as a linear map from $W^{\max}$ into W), then $\bigcup_\tau \mathrm{Supp}(\tau) = \mathrm{Spec} = \widehat{W}$ exhaust all representations of W.

The case of surjective induction was also paid special attention in [1]. Indeed, there are many other cases when $\bigcup_\tau \mathrm{Supp}(\tau) = \mathrm{Spec}$, but we have no satisfactory characterization yet.

Proof: We want to compute the σ-support $\mathrm{Supp}(\mathrm{id}_{A^{\max}})$. For this purpose, we multiply $\phi(\sigma_{\mathrm{id}_{A^{\max}}})$ with $\phi(\mu)$, $\mu \in W$. Using in turn α-σ-reciprocity, the representation condition for ϕ, Frobenius reciprocity, the homomorphism property of α-induction, and associativity of fusion, we arrive at

$$\begin{aligned} \phi(\sigma_{\mathrm{id}_{A^{\max}}})\phi(\mu) &= \sum_\lambda \phi(\lambda)\phi(\mu)\langle\alpha_\lambda, \mathrm{id}_{A^{\max}}\rangle = \sum_{\kappa\lambda} N^\lambda_{\kappa\bar{\mu}}\phi(\kappa)\langle\alpha_\lambda, \mathrm{id}_{A^{\max}}\rangle = \\ &= \sum_\kappa \phi(\kappa)\langle\alpha_\kappa\bar{\alpha}_\mu, \mathrm{id}_{A^{\max}}\rangle = \sum_{\kappa,\beta} \phi(\kappa)\langle\bar{\alpha}_\mu, \beta\rangle\langle\alpha_\kappa\beta, \mathrm{id}_{A^{\max}}\rangle. \end{aligned}$$

Here the sum over β extends over all sectors of V. The last sum must vanish for every μ, since the left hand side does, if $\phi(\sigma_{\mathrm{id}_{A^{\max}}}) = 0$, i.e., if $\phi \notin \mathrm{Supp}(\mathrm{id}_{A^{\max}})$. Now, if α-induction is surjective, then every sector β arises as a linear combination of sectors α_μ, and consequently

$$\sum_\kappa \phi(\kappa)\langle\alpha_\kappa\beta, \mathrm{id}_{A^{\max}}\rangle = \sum_\lambda \overline{\phi(\lambda)}\langle\alpha_\lambda, \beta\rangle$$

must vanish for all β. These are sufficiently many matrix entries to ensure the vanishing of the full matrix (since $\langle\alpha\beta_1, \beta_2\rangle = \sum_\beta\langle\alpha, \beta\rangle\langle\beta\beta_1, \beta_2\rangle$), and hence the absence of ϕ from the α-spectrum Spec. Hence $\mathrm{Spec} \subset \mathrm{Supp}(\mathrm{id}_{A^{\max}})$, implying the first claim.

On the other hand, if σ-restriction is surjective, then $\phi(\sigma_\tau) = 0$ for all $\tau \in W^{\max}$ implies $\phi(\lambda) = 0$ for all $\lambda \in W$, hence $\phi = 0$. Thus the union of the σ-supports exhausts all representations of W, implying the second claim. $\square$

We have thus established some first constraints on the coupling matrix in terms of representations of fusion algebras.

Further constraints are expected to derive from locality which was only partially exploited in the form of α-σ-reciprocity in Proposition 4.1, and in the commutativity of left and right chiral observables in Theorem 3.6. Notably the condition for locality of the 2D theory in terms of the local subfactor data and the statistics which was given in [17] remains to be transcribed into a condition on the coupling matrix.

As mentioned in the introduction, chiral locality produces matrices $S^{\rm stat}$ and $T^{\rm stat}$ which represent $\mathrm{SL}(2,\mathbb{Z})$ [27, 6], except for a possible degeneracy of the braiding. A first implication of the locality condition for the 2D theory is that $T_L^{\rm stat} Z = Z T_R^{\rm stat}$, in accordance with local 2D conformal fields having integer spin $h_L - h_R$. The companion relation $S_L^{\rm stat} Z = Z S_R^{\rm stat}$, that is, modular invariance of the coupling matrix with respect to the representation of $\mathrm{SL}(2,\mathbb{Z})$ given by the statistics, cannot be established for general 2D nets B, however. The surprise is that, as shown here, one can go much of the way towards classification without knowing these formulae, and that one can do so whether the involved sectors have a degenerate braiding or not. (Müger's proof [21] that the degeneracy can always be removed by an algebraic extension of the chiral observables does not help here, since this extension is in general not possible within the given 2D observables.)

5 Conclusions

We have shown that in a 2D conformally invariant quantum field theory with sufficiently many chiral observables to generate the chiral Möbius groups, there are maximal algebras of chiral observables which are, locally, the relative commutants of each other, as well as of any a priori given chiral observables sharing the generating property (cf. Sect. 2).

The representation theory of the chiral observables is governed by a "canonical tensor product subfactor" (CTPS) $A_L \otimes A_R \subset B$ given by the respective chiral and 2D local algebras. We have therefore investigated the general structure of CTPS's and have found a characterization of the two tensor factors being each other's relative commutants ("normality") in terms of a coupling matrix. The coupling matrix in this case provides an isomorphism between the respective fusion rules for the involved sectors of the two tensor factors.

This abstract result, applied to the quantum field theoretical situation at hand, generalizes a statement on certain "extended" chiral observables in the classification program for 2D modular invariant partition functions, and shows that the latter coincide with the maximal chiral observables.

Exploiting general properties of α-induction and σ-restriction between the superselection sectors of the maximal and the a priori given non-maximal chiral observables, constraints on the coupling matrix (with respect to the non-maximal chiral observables) are derived which are the direct counterparts of similar constraints in the modular classification program.

Yet, modular invariance has not been assumed throughout the analysis. This supports our conviction that modular invariants are just one aspect of a deeper and more general mathematical structure (presumably related to "asymptotic subfactors" and "quantum doubles"). A classification in terms of graphs still remains to be established in the general situation. Possibly, additional constraints originating from locality will play a role here.

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