MEASURE RIGIDITY FOR THE ACTION OF SEMISIMPLE GROUPS IN POSITIVE CHARACTERISTIC

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Abstract. We classify measures on a homogeneous space which are invariant under a certain solvable subgroup of a semisimple subgroup and are ergodic under the unipotent radical of this subgroup in arbitrary characteristic. As a corollary we get a measure classification for the action of semisimple subgroups without characteristic restrictions.

1. Introduction

Ratner’s celebrated work, [R90b, R92, R95] see also [MT94], classifies all probability measures invariant and ergodic under a one parameter unipotent subgroup in the setting of homogeneous spaces over local fields of characteristic zero. In positive characteristic setting, however, classification theorems in this generality are not yet available.

It is plausible that if one assumes that the characteristic is “large”, then one can carry out the proof in [MT94] to this setting; without such characteristic restrictions, however, the situation is more subtle. Roughly speaking, the main technical difficulty arises from the fact that the image of a polynomial map over a field of positive characteristic can lie in a proper subfield, and hence the image may be “small”. This simple fact enters our study as follows. The divergence of the orbits of two nearby points under the unipotent group is governed by a certain polynomial like map, see 4. Now slow growth of polynomial maps and Birkhoff ergodic theorem imply that $\mu$ is invariant under the image of this map, see 4.3. This important property of unipotent flows plays a crucial role in the argument. Indeed using this one can increase the “dimension” of the group leaving $\mu$ invariant. In the positive characteristic setting this construction only guarantees the dimension is increase by a (non-constant) fraction at each step, thus, there is no a priori reason for this process to terminate.

In recent years there have been some partial results in this direction, [M11, EM12]. In particular, the action of semisimple groups was considered in [EG10]. It is worth mentioning that the results in [EG10] are proved under the assumption that the

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characteristic is “large”. Our account here, in particular, removes the characteristic restriction from the main result in [EG10].

This generalization introduces serious technical difficulties. These difficulties require a comprehensive use of the main ideas and techniques from [MT94]; the proof in [EG10], however, relies on a simpler argument which goes back to [E06]. We also borrow extensively from the theory of algebraic groups in this work, this also seems inevitable in this setting.

We need some notation in order to state the main results of the paper. Let \( T \) be a finite set, and for any \( v \in T \) let \( k_v \) be a local field; put \( k_T = \prod_{v \in T} k_v \). For any \( v \in T \) let \( G_v \) be a \( k_v \)-algebraic group and let \( G_v = G_v(k_v) \). Define \( G = \prod_{v \in T} G_v \), and \( G = G(k_T) \), and let \( \Gamma \) be a discrete subgroup of \( G \).

We now describe a certain family of solvable subgroups in \( G \) to which our main theorem applies. Such considerations are inevitable in positive characteristic setting; indeed, many subgroups only have algebraic description after changing the ground field to a finite subfield, which may very well be inseparable, see §3. An important special case of the following discussion is obtained by taking \( H = \text{SL}_2(k'_w) \subset G_w \) where \( k'_w \) is a closed subfield of \( k_w \) for some \( w \in T \). Then letting \( SU \) be the group of upper triangular matrices in \( H \), see Corollary 1.2.

Fix an element \( w \in T \) once and for all and let \( k' \) be a closed subfield of \( k_w \). Suppose \( k''/k' \) is a finite extension of \( k' \), note that \( k'' \) is not necessarily a subfield of \( k_w \). Let \( \mathbb{H}' \) be a connected \( k'' \)-almost simple, \( k'' \)-group which is isotropic over \( k'' \), and put \( \mathbb{H} = \mathcal{R}_{k''/k'}(\mathbb{H}') \), where \( \mathcal{R} \) denotes Weil’s restriction of scalars, see Definition 2.4. In particular, we have \( \mathbb{H}(k') = \mathbb{H}'(k'') \). It is worth mentioning that when \( k''/k' \) is an inseparable extension, \( \mathcal{R}_{k''/k'}(\mathbb{H}') \) is no longer a reductive group. It has a unipotent radical which is \( k' \)-closed but not defined over \( k' \), see Definition 2.2. Such groups are examples of pseudo reductive groups. We refer to [CGP10] for a comprehensive study and the structure theory of pseudo reductive groups, our account here, however, does not require the knowledge of this subject.

Fix a non central cocharacter \( \lambda \) of \( \mathbb{H} \), that is \( \lambda : \mathbb{G}_m \to \mathbb{H} \) is non central homomorphism defined over \( k' \); such homomorphism exists thanks to the fact that \( \mathbb{H}' \) is \( k'' \)-isotropic. Put \( S = \lambda(\mathbb{G}_m) \). Let \( s' \in S(k') \) be a nontrivial element and set \( U = \mathcal{W}_{\mathbb{H}}^+(s') \), see [2] for the notation.

Throughout the paper, we fix a \( k_w \)-homomorphism \( \iota : \mathbb{H} \to \mathbb{G}_w \) with finite kernel, and put \( H = \iota(\mathbb{H}(k')) \), \( S = \iota(S(k')) \), and \( U = \iota(U(k')) \).

We recall the following definition. Let \( M \) be a locally compact second countable group and suppose \( \Lambda \) is a discrete subgroup of \( M \). A probability measure \( \mu \) on \( M/\Lambda \) is called homogeneous if there exists \( x \in M/\Lambda \) such that \( \Sigma x \) is closed, and \( \mu \) is the \( \Sigma \)-invariant probability on \( \Sigma x \) where \( \Sigma \) is the closed subgroup of all elements of \( M \) which preserve \( \mu \).
Theorem 1.1. Let \( \mu \) be a probability measure on \( G/\Gamma \) which is invariant under the action of \( SU \) and is \( U \)-ergodic. Then \( \mu \) is a homogeneous measure.

The following corollary presents an interesting special case. Let \( k' \subset k_w \) be a closed subfield as above and let \( \iota: \text{SL}_2 \times_k k_w \to \mathbb{G}_w \) be a \( k_w \)-homomorphism with finite kernel. Let \( H = \iota(\text{SL}(k')) \subset G \). Then

**Corollary 1.2.** If \( \mu \) is an \( H \)-invariant ergodic measure on \( G/\Gamma \), then \( \mu \) is a homogeneous measure.

**Proof.** Let \( SU \) be the image of the group of upper triangular matrices in \( \text{SL}(k') \). Then \( \mu \) is \( SU \)-invariant and \( U \)-ergodic, see [Mar90b, Lemma 3.4]. Therefore, the corollary follows from Theorem 1.1. \( \square \)

Indeed the group \( \Sigma \) in the definition of homogeneous measure will have an “algebraic description”. The statement of this refinement involves definitions and notation which will be developed later, thus, we postpone the statement and the proof of this refinement to Theorem 5.9.

Our general strategy of the proof is similar to the strategy in [MT94]. We construct extra invariance for the measure using the construction of quasi-regular maps, see §4 and utilize entropy. However, several technical difficulties arise as we described above. We overcome the main technical difficulty, i.e. polynomial maps may have “small” image, by using the action of \( S \). Roughly speaking, if the measure is invariant under an element which does not centralize \( S \), then we can conjugate by \( S \) and obtain invariance under image of a map whose degree may be controlled in terms of the weights of \( S \). This is made precise in Proposition 3.1, which is of independent interest. Using this fact we get a control on the structure of the measure along the contracting leaves, which is essential in order to utilize entropy, see Proposition 5.3 and Corollary 5.4.

We close the introduction with the following remark. In view of the Mautner phenomena, any \( SU \)-ergodic measure is \( S \)-ergodic. However, we have made a stronger assumption that \( \mu \) is \( U \)-ergodic. It is worth mentioning that the \( U \)-ergodicity of \( \mu \) is used in a crucial way. Indeed if \( \mu \) is not \( U \)-ergodic, then the Basic Lemma in §4 provides “new” group elements which a priori only leave a \( U \)-ergodic component of \( \mu \) invariant. However, \( S \) does not need to normalize the invariance group of a \( U \)-ergodic component. Therefore, the above mentioned use of the action of \( S \) through Proposition 3.1 is not available.

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## 2. Notation and Preliminary Lemmas

Let \( T \) be a finite set and let \( k_v \) be a local field for all \( v \in T \); define \( k_T = \prod_{v \in T} k_v \) as in the introduction. We endow \( k_T \) with the norm \( | \cdot | = \max_{v \in S} | \cdot |_v \) where \( | \cdot |_v \) is a norm on \( k_v \) for each \( v \in T \).
2.1. \(k_T\)-algebraic groups. A \(k_T\)-algebraic group \(M\) (resp. \(k_T\)-algebraic variety \(\overline{M}\)) is the formal product of \(\prod_{v \in T} M_v\) of \(k_v\) algebraic groups (resp. \(\prod_{v \in T} \overline{M}_v\) of \(k_v\) algebraic varieties). The usual notions from elementary algebraic geometry e.g. regular maps, rational maps, rational point etc. are defined componentwise. We will take this as to be understood, and use these notions without further remarks. There are two topologies on \(M(k_T)\), the Hausdorff topology and the Zariski topology. We will make this clear when referring to the Zariski topology. Hence, if in a topological statement we do not give reference to the particular topology used, then the one which is being considered is the Hausdorff topology.

Let \(M\) be a \(k_T\)-group. An element \(e \neq g \in M(k_T)\) is an element of class \(A\) if \(g = (g_v)_{v \in T}\) is diagonalizable over \(k_T\), and for all \(v \in T\) the component \(g_v\) has eigenvalues which are integer powers of the uniformizer \(\pi_v\) of \(k_v\).

Given a subset \(B \subset M(k_T)\) we let \(\langle B \rangle\) denote the closed (in the Hausdorff topology) group generated by \(B\).

2.2. Pseudo-parabolic subgroups. Let \(k\) be a local field. Suppose \(M\) is a connected \(k\)-algebraic group, and let \(\lambda : G_m \to M\) be a noncentral homomorphism defined over \(k\). Define \(-\lambda(a) = \lambda(a)^{-1}\) for all \(a \in k^*\). As in [Sp98.13.4] and [CGP10.13.4] and [BT78], we let \(P_M(\lambda)\) denote the closed subgroup of \(M\) formed by those elements \(x \in M\), such that the map \(\lambda(a)x\lambda(a)^{-1}\) extends to a map from \(G_a\) into \(M\).

Let \(Z_M^+ (\lambda)\) be the normal subgroup of \(P_M(\lambda)\), formed by \(x \in P(\lambda)\) such that \(\lambda(a)x\lambda(a)^{-1} \to e\) as \(a\) goes to \(0\). The centralizer of the image of \(\lambda\) is denoted by \(Z_M(\lambda)\). Similarly define \(W_M(\lambda)\) which we will denote by \(W_M^-(\lambda)\).

The multiplicative group \(G_m\) acts on \(\text{Lie}(M)\) via \(\lambda\), and the weights are integers. The Lie algebras of \(Z_M(\lambda)\) and \(W_M^+(\lambda)\) may be identified with the weight subspaces of this action corresponding to the zero, positive and negative weights. It is shown in [CGP10.13.4] and [BT78], that \(P_M(\lambda)\), \(Z_M(\lambda)\) and \(W_M^+(\lambda)\) are \(k\)-subgroups of \(M\). Moreover, \(W_M^+(\lambda)\) is a normal subgroup of \(P_M(\lambda)\) and the product map

\[ Z_M(\lambda) \times W_M^+(\lambda) \to P_M(\lambda) \text{ is a } k\text{-isomorphism of varieties.} \]

A pseudo-parabolic \(k\)-subgroup of \(M\) is a group of the form \(P_M(\lambda)R_{\alpha,k}(M)\) for some \(\lambda\) as above where \(R_{\alpha,k}(M)\) denotes the maximal connected normal unipotent \(k\)-subgroup of \(M\). [CGP10, Def. 2.2.1].

We also recall from [CGP10 Prop. 2.1.8(3)] that the product map

\[ W_M^-(\lambda) \times Z_M(\lambda) \times W_M^+(\lambda) \to M \text{ is an open immersion of } k\text{-schemes.} \]

It is worth mentioning that these results are generalization to arbitrary groups of analogous and well known statements for reductive groups.

Let \(M = \mathbb{M}(k)\), and put

\[ W_M^+(\lambda) = W_M^+(\lambda)(k), \text{ and } Z_M(\lambda) = Z_M(\lambda)(k). \]

From Proposition 2.1 we conclude that \(W_M^-(\lambda)Z_M(\lambda)W_M^+(\lambda)\) is a Zariski open dense subset of \(M\), which contains a neighborhood of identity with respect to the Hausdorff topology.
For any $\lambda$ as above define
\begin{equation}
M^+(\lambda) := \langle W^+_M(\lambda), W^-_M(\lambda) \rangle.
\end{equation}

**Lemma 2.1.** The group $M^+(\lambda)$ is a normal subgroup of $M$ for any $\lambda$ as above. Moreover, $M^+(\lambda)$ is unimodular.

**Proof.** Since $D := W^-_M(\lambda)Z_M(\lambda)W^+_M(\lambda)$ is a Zariski open dense subset of $M$, we have $DD = M$. In particular, $\langle W^-_M(\lambda), Z_M(\lambda), W^+_M(\lambda) \rangle = M$. This together with the fact that $W^+_M(\lambda)$ is normalized by $Z_M(\lambda)$ implies the first claim.

We now show that $M^+(\lambda)$ is unimodular. First note that, in view of the relations between the Haar measure and algebraic form of top degree, see [Bour 10.1.6] and [Oe84, Thm. 2.4], we have
\begin{equation}
\Delta_M(g) = |\wedge^\text{dim} M \text{Ad}(g) \wedge^\text{dim} M \text{Lie}(M)|
\end{equation}
where $\Delta$ is the modular function. Since $W^\pm_M(\lambda)$ are unipotent subgroups, we get from the above that $\Delta_M(g) = 1$ for all $g \in W^+_M(\lambda)$. Hence
\begin{equation}
\Delta_M(g) = 1 \text{ for all } g \in M^+(\lambda).
\end{equation}

In view of the first claim, $M^+(\lambda)$ is a normal subgroup of $M$, therefore, $M/M^+(\lambda)$ is a locally compact group. In particular, it has an $M/M^+(\lambda)$-invariant measure. This implies that $M/M^+(\lambda)$ has an $M$-invariant measure. Hence, thanks to [Rag72, Lemma 1.4], we get that
\begin{equation}
\Delta_M^+(\lambda)(g) = \Delta_M(g) \text{ for all } g \in M^+(\lambda).
\end{equation}

This implies the claim in view of (2.3). \hfill \Box

Given an element $s \in M$ from class $A$. There is $\lambda : \mathbb{G}_m \to M$ so that $s = \lambda(a)$ for some $a \in k$ with $|a| > 1$. Then we define
\begin{equation}
W^\pm_M(s) := W^\pm(\lambda).
\end{equation}

2.3. When working with algebraic groups over a non perfect field $k$, say with characteristic $p > 0$, it is convenient to use the language of group schemes. Indeed certain natural objects, e.g. kernel of a $k$-morphism, is not necessarily defined over the base field as linear algebraic groups in the sense of [B91] or [Sp98]. They are so called “$k$-closed”, i.e. they are closed group schemes over the base field, however, they are not necessarily smooth group schemes. We have tried to avoid this language, and have tried to work in the more familiar setting of linear algebraic groups. The draw back, however, is that we will need to use the notation of a $k$-closed set whose definition we now recall. Let $K$ be an algebraically closed field which contains $k$.

**Definition 2.2.** ([B91, AG, §12.1]) Let $\mathcal{M} = \text{Spec}(K[x_1, \ldots, x_n]/I)$, where $I$ is the ideal of all functions vanishing on $\mathcal{M}$. If the ideal $I$ is defined over $k$, then the set $\mathcal{M}$ is called $k$-closed. \hfill \Box

\footnote{Let us remark that over a perfect field the notation of $k$-closed and that of a variety defined over $k$ coincide.}
If $\mathbb{M}$ is $k^{p^{-\infty}}$-closed, then it is also $k$-closed. That is to say: $k$-topology and $k^{p^{-\infty}}$-topology coincide. Let us mention that if $M \subset k^n$ is a set which is the zero set of an ideal $I$ in $k[x_1, \cdots, x_n]$, then $M$ is the $k$-points of a $k$-closed set. This is how the $k$-closed sets will arise in our study. Abusing the notion a subset of $k^n$ will be called $k$-closed if it may be realized as the $k$-points of a $k$-closed subset of $M^n$. A $k$-closed set is defined over $k$ as a Scheme but not as a variety. What is useful fact is however that if we start we a subset of $k$-points of a variety and take the Zariski closure of this set, then we get a variety defined over $k$. The following is the precise formulation and generalization of this fact.

**Lemma 2.3** (Cf. [CGP10], Lemma D.3.1). Let $\mathbb{M}$ be a scheme locally of finite type over a field $k$. There exists a unique geometrically reduced closed subscheme $\mathbb{M}' \subset \mathbb{M}$ such that $\mathbb{M}'(k') = \mathbb{M}(k')$ for all separable field extensions $k'/k$. The formation of $\mathbb{M}'$ is functorial in $\mathbb{M}$, and commute with the formation of products over $k$ and separable extension of the ground field. In particular, if $\mathbb{M}$ is a $k$-group scheme, then $\mathbb{M}'$ is a smooth $k$-subgroup scheme.

Let us also recall the definition of the Weil restriction of scalars.

**Definition 2.4.** Let $k$ be a field and $k'$ a subfield of $k$ such that $k/k'$ is a finite extension, and let $\mathbb{M}$ be an affine $k$-variety. The Weil restriction of scalars $R_{k/k'}(\mathbb{M})$ is the affine $k'$-scheme satisfying the following universal property

\begin{equation}
R_{k/k'}(\mathbb{M})(B) = \mathbb{M}(k \otimes_{k'} B)
\end{equation}

for any $k'$-algebra $B$.

### 2.4. Ergodic measures on algebraic varieties

Let $\mathbb{M}$ be a $k_T$-algebraic group and let $M = \mathbb{M}(k_T)$. Suppose $B \subset M$ is a group which is generated by one parameter $k_T$-split unipotent algebraic subgroups and by one dimensional $k_T$-split tori. Let $\Lambda$ be a discrete subgroup of $M$ and put $\pi : M \to M/\Lambda$ to be the natural projection. We have the following.

**Lemma 2.5** (Cf. [MT94], Proposition 3.2). Let $\mu$ be a $B$-invariant and ergodic Borel probability measure on $M/\Lambda$. Suppose $\mathbb{D}$ is a $k_T$-closed subset of $\mathbb{M}$ and put $D = \mathbb{D}(k_T)$. If $\mu(\pi(D)) > 0$, then there exists a connected $k_T$-algebraic subgroup $E$ of $\mathbb{M}$ such that $B \subset E := E(k_T)$, and a point $g \in D$ such that $Eg \subset D$ and $\mu(\pi(Eg)) = 1$. Moreover, $E \cap g\Lambda g^{-1}$ is Zariski dense in $E$.

**Proof.** First note that thanks to Lemma 2.3 we may assume that $D$ is the $k_T$-points of a $k_T$-variety. Now since the Zariski topology is Noetherian we may and will assume that $\mathbb{D}$ is minimal $k_T$-variety in the sense that $\mu(\pi(\mathbb{D}'(k_T))) = 0$ for all proper $k_T$-varieties $\mathbb{D}' \subset \mathbb{D}$. This, in particular, implies that $D$ is irreducible.

Put $B' := \{g \in B : \pi(gD) = \pi(D)\}$. The minimality assumption implies that $\mu(\pi(D) \setminus \pi(gD)) = 0$ for all $g \notin B'$. Since $\mu$ is a probability measure, we get that $B'$ has finite index in $B$. Let now $g \in B'$ then $gD \subset DA$, and since $\Lambda$ is countable we get that there exists some $\lambda \in \Lambda$ so that $\mu(\pi(gD \cap D\lambda)) > 0$. Using our minimality assumption one more time, we get that $gD \subset DA$. This implies $gD = DA$. Since $\Lambda$ is countable we get that $B'/B''$ is countable where $B'' := \{g \in B' : gD = D\}$. All together we have $B''$ has countable index in $B$. Recall now that $B = \langle B_i \rangle$ where each $B_i = B_i(k_T)$ and $B_i$ is either a one parameter $k_T$-split unipotent algebraic
subgroup, or a one dimensional $k_T$-split tori. In particular, $B_i$ is connected for all $i$. This implies $B_i'' = B_i \cap B''$ is Zariski dense in $B_i$. Hence $B_i D = D$ for all $i$, which implies $BD = D$.

Put $\Lambda_0 = \{ \lambda \in \Lambda : D \lambda = D \}$, and $Y = D \setminus \cup_{\Lambda \in \Lambda_0} D \lambda$. Minimality of $D$ implies that $\mu(\pi(Y)) = \mu(\pi(D))$. Moreover, $BY \Lambda_0 = Y$ and the natural map $Y/\Lambda_0 \to M/\Lambda$ is injective. We thus get a $B$-invariant ergodic probability measure $\mu_0$ on $Y/\Lambda_0$. Let $F$ be the Zariski closure of $\Lambda_0$ in $M$ then $D F = \emptyset$. The push forward of $\mu_0$ gives a $B$-ergodic invariant measure on $D/F \subseteq (D/F)(k_T)$ where $F = F(k_T)$. Now by Shl99 Thm. 1.1 and 3.6], this measure is the Dirac mass at one point. That is there is some $z \in D$ so that $\mu(\pi(z F)) = 1$. Since $z F \subseteq D$, our minimality assumption implies $z F = D$. Hence $g = z$ and $E = g F g^{-1}$ satisfy the claims in the lemma. □

Let the notation be as in the beginning of §2.4 We will say a Borel probability measure $\mu$ on $M/\Lambda$ is Zariski dense if there is no proper $k_T$-closed subset $M \subseteq M$ such that $\mu(\pi(M)) > 0$, where $M = M(k_T)$. Two $k_T$-subvarieties $L_1$ and $L_2$ of $M$ are said to be transverse at $x$ if they both are smooth at $x$ and

$$T_x(L_1) \oplus T_x(L_2) = T_x(M),$$

where $T_x(\bullet)$ denotes the tangent space of $\bullet$ at $x$. Thanks to Lemma 2.3 we also have the following, see MT94 Prop. 3.3.

Lemma 2.6. Suppose $B = B(k_T)$ for a $k_T$-subgroup $B$ of $M$. Assume that $\mu$ is a Zariski dense $B$-invariant Borel probability measure on $M/\Lambda$. Suppose $L$ is a connected $k_T$-algebraic subvariety of $M$ containing $e$ which is transverse to $B$ at $e$. Let $D \subseteq L$ be a $k_T$-closed subset of $L$ containing $e$. Then, there exists a constant $0 < \varepsilon < 1$ so that the following holds. If $\Omega \subset M/\Lambda$ is a measurable set with $\mu(\Omega) > 1 - \varepsilon$, then one can find a sequence $\{g_n\}$ of elements in $M$ with the following properties

(i) $\{g_n\}$ converges to $e$,
(ii) $g_n \Omega \cap \Omega \neq \emptyset$, and
(iii) $\{g_n\} \subset L(k_T) \setminus D(k_T)$.

2.5. Homogeneous measures. Let $M$ be a locally compact second countable group and let $\Lambda$ be a discrete subgroup of $M$. Suppose $\mu$ is a Borel probability measure on $M/\Lambda$. Let $\Sigma$ be the closed subgroup of all elements of $M$ which preserve $\mu$. The measure $\mu$ is called homogeneous if there exists $x \in M/\Lambda$ such that $\Sigma x$ is closed, and $\mu$ is the $\Sigma$-invariant probability measure on $\Sigma x$.

Lemma 2.7. (cf. MT94 Lemma 10.1]) Let $M$ be a locally compact second countable group and $\Lambda$ a discrete subgroup of $M$. Suppose $B$ is a normal and unimodular subgroup of $M$. Then any $B$-invariant $B$-ergodic measure on $M/\Lambda$, is homogeneous. Moreover, $\Sigma = B \Lambda$.

2.6. Modulus of conjugation. Suppose $M$ is a $k_T$-algebraic group. Then $M = M(k_T)$ is a locally compact group. Let $a \in M$ and suppose $B \subset M$ is a closed subgroup of which is normalized by $a$. We let $\alpha(a, B)$ denote the modulus of the conjugation action of $a$ on $B$, i.e. if $Y \subset B$ is a measurable set

$$\theta(a Ya^{-1}) = \alpha(a, B) \theta(Y)$$
where $\theta$ denotes a Haar measure on $B$.

Note that $\alpha(a,M) = 1$ if $a \in [M,M]$; in particular, if $A$ is a subgroup of $M$ such that $[A,A] = A$, then $\alpha(a,M) = 1$ for all $a \in A$.

3. An algebraic statement

One of the remarkable features of Ratner’s theorems is that they connect objects which are closely connected to the Hausdorff topology of the underlying group, like closure of a unipotent orbit or a measure invariant by a unipotent group, to objects which are described using the Zariski topology, e.g. algebraic subgroups. In positive characteristic setting these two topologies are rather far from each other. From a philosophical standpoint, this is one reason why the existing proofs in characteristic zero do not easily generalize to this case.

Let us restrict ourselves to unipotent groups in order to highlight one of the differences. In characteristic zero, the group generated by one unipotent matrix already carries quite a lot of information, e.g. it is Zariski dense in a one dimensional group. In positive characteristic, however, all unipotent elements are torsion. The situation improves quite a bit in the presence of a split torus action. In a sense, such an action can be used “to redefine a notion of Zariski closure” for the group generated by one element. This philosophy is used in this section where we prove the following proposition which is of independent interest.

**Proposition 3.1.** Let $k$ be a local field of characteristic $p > 0$ and $W$ a unipotent $k$-group equipped with a $k$-action by $GL_1$. Assume that all the weights are positive integers. Let $U$ be a subgroup of $W(k)$ which is invariant under the action of $k^\times$. Then, there is $q$ a power of $p$, which only depends on the set of weights, such that $U = W'(k^q)$, where $W'$ is the Zariski-closure of $U \subset W(k) = R_{k/k^q}(W)(k^q)$ in $R_{k/k^q}(W)$. Moreover, $W'$ is connected.

Groups like $U$ arise naturally in our study, see §5 for more details.

We shall prove Proposition 3.1 in several steps. First we prove it when $W$ is a commutative $p$-torsion $k$-group. In the next step, the general commutative case is handled. In the final step, the general case is proved by induction on the nilpotency length.

In order to prove the first step, we shall start with a few auxiliary lemmas. In Lemmas 3.2 and 3.3, we assume that the weights are powers of $p$. In Lemmas 3.4 and 3.5, we get a convenient decomposition of $U$ into certain subgroups, and finally in Lemma 3.6 we prove the first step. Let begin with the following

**Lemma 3.2.** Let $F$ be an infinite field of characteristic $p$ and $0 < l_1 < \cdots < l_n$ positive integers. Assume $GL_1$ acts linearly on a standard $F$-vector group $W$ with weights equal to $p^{l_i}$. Let $W_i$ be the weight space of $p^{l_i}$ and suppose $U$ is a subgroup of $W(k)$ which is invariant under $GL_1(k)$. If $U$ does not intersect $\bigoplus_{i=2}^n W_i$, then $U = W'(F^{p^{l_1}})$, for any $l \geq l_1$, where $W'$ is the Zariski-closure of $U$ in $R_{F/F^{p^{l_1}}}(W)$.

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2 See $[P98]$ where structure of compact subgroups of semisimple groups of adjoint type is described.
Proof. Via the action of $\text{GL}_1(F)$, and in view of our assumption of the weights, we can view $\mathbb{W}(F)$ as an $F$-vector space and $U$ as a $F$-subspace. Since $U$ does not intersect $\bigoplus_{i=2}^n \mathbb{W}_i$, we get an $F$-linear map $\theta$ from $\text{pr}_1(U)$ to $\bigoplus_{i=2}^n \mathbb{W}_i$, where \( \text{pr}_1 : \mathbb{W} \to \mathbb{W}_1 \) is the projection map, and we have
\[
U = \{(x, \theta(x)) | x \in \text{pr}_1(U)\}.
\]
It is clear that $\text{pr}_1(U)$ is a $F^{p_1}$-subspace of $\mathbb{W}_1(F)$ with respect to the standard scalar multiplication. It is also clear that $\theta$ can be extended to an $F$-morphism from $\mathbb{W}_1$ to $\bigoplus_{i=2}^n \mathbb{W}_i$. Hence there is a standard $F^{p_1}$-vector subgroup $\mathbb{W}'_1$ of $\mathcal{R}_{F/F^{p_1}}(\mathbb{W}_1)$ such that
\[
U = \{(x, y) | x \in \mathbb{W}'_1(F^{p_1}), y = \mathcal{R}_{F/F^{p_1}}(\theta)(x)\}.
\]
Since $F$ is an infinite field, the Zariski-closure $\mathbb{W}'$ of $U$ in $\mathcal{R}_{F/F^{p_1}}(\mathbb{W})$ is equal to
\[
\{(x, y) | x \in \mathbb{W}'_1, y = \mathcal{R}_{F/F^{p_1}}(\theta)(x)\},
\]
which shows that $U = \mathbb{W}'(F^{p_1})$. Now, one can easily deduce the same result for any $l \geq l_1$. \(\square\)

Lemma 3.3. Let $F$ be an infinite field of characteristic $p$ and $0 < l_1 < \cdots < l_n$ positive integers. Assume $\text{GL}_1$ acts linearly on a standard $F$-vector group $\mathbb{W}$ with weights equal to $p^{l_i}$. Let $U$ be a subgroup of $\mathbb{W}(F)$ which is invariant under $\text{GL}_1(F)$. Then
\[
U = \mathbb{W}'(F^{p_{l_i}}),
\]
for any $l \geq l_n$, where $\mathbb{W}'$ is the Zariski-closure of $U$ in $\mathcal{R}_{F/F^{p_{l_i}}}(\mathbb{W})$.

Proof. We denote the weight space of $p^{l_i}$ by $\mathbb{W}_i$. Let $U_i = U \cap (\bigoplus_{j=1}^n \mathbb{W}_j)$ and define $U_i'$ to be a $\text{GL}_1(F)$-invariant complement of $U_i$ in $U_{i-1}$. So we have
\[
U_i' \subset \bigoplus_{j=1}^n \mathbb{W}_j,
\]
\[
U_i' \cap (\bigoplus_{j=1}^n \mathbb{W}_j) = \{0\},
\]
\[
U_{i-1} = U_i \oplus U_i',
\]
\[
U = U_i' \oplus U_i + \cdots \oplus U_i' \oplus U_n.
\]
By Lemma 3.2, we have that $U_i' = \mathbb{W}_i'(F^{p^{l_i}})$, for all $i$ and any $l \geq l_{i-1}$, where $\mathbb{W}_i'$ is the Zariski-closure of $U_i'$ in $\mathcal{R}_{k/k^{p_{l_i}}}((\bigoplus_{j=1}^n \mathbb{W}_j))$. Moreover, $U_n$ is a subspace of $\mathbb{W}_n$ with respect to the standard action of $F^{p_{l_n}}$; one can easily conclude. \(\square\)

Lemma 3.4. Let $m_1, \ldots, m_d$ be distinct positive integers which are coprime with $p$. Let $g(x) = (x^{m_1}, \ldots, x^{m_d})$ be a morphism from $\mathbb{A}^1$ to $\mathbb{A}^d$. Then
\[
G(x_1, \ldots, x_d) := g(x_1) + \cdots + g(x_d)
\]
is a separable function from $\mathbb{A}^d$ to $\mathbb{A}^d$ at a $F$-point, for any infinite field $F$ of characteristic $p$.

Proof. It is enough to show that the Jacobian of $G$ is invertible at some $F$-point. Thus, thanks to our assumption: all of $m_i$ are co-prime with $p$, it suffices to show that the kernel of $D = [x_j^{m_i-1}]$ is trivial for some $x_j \in F$. Now since $F$ is an infinite field, there is an element $x \in F^\times$ of multiplicative order larger than max $m_i$. Set
$x_j = x_j^{d-1}$; if $D$ has a non-trivial kernel, then there is non-zero polynomial $Q$ of degree at most $d-1$ with coefficients in $F$, such that
\[ Q(x_1^{m_1-1}) = Q(x_2^{m_2-1}) = \cdots = Q(x_\ell^{m_\ell-1}) = 0. \]
This is a contradiction as $x_\ell^{m_\ell-1}$ are distinct and the degree of $Q$ is at most $d-1$. □

**Lemma 3.5.** Let $k$ be a local field of characteristic $p$ and $m_1, \ldots, m_\ell$ distinct positive integers which are coprime with $p$. Let $W$ be a standard $k$-vector group equipped with a linear $k$-action by $GL_1$. Assume that the set of weights $\Phi = \Phi_1 \cup \cdots \cup \Phi_\ell$, $\Phi_i = \{p^{\ell_1}m_1, p^{\ell_2}m_2, \ldots, p^{\ell_\ell}m_\ell\}$ and moreover $\Phi_i$ is non-empty. Let $U$ be a subgroup of $W(k)$, which is invariant under the action of $GL_1(k)$. Then $U = U_1 \oplus \cdots \oplus U_\ell$, where $U_i = U \cap (\bigoplus_{\alpha \in \Phi_i} W_\alpha)$ and $W_\alpha$ is the weight space corresponding to $\alpha$. Furthermore, if $\Phi_i = \{p^{\ell_1}m_1, p^{\ell_2}m_2, \ldots, p^{\ell_\ell}m_\ell\}$ and $x = (x_1, \ldots, x_\ell) \in U_i$, then
\[ (\lambda^{\ell_1}x_1, \ldots, \lambda^{\ell_\ell}x_\ell) \in U_i, \text{ for any } \lambda \in k^\times. \]

**Proof.** Take an arbitrary element $x = (x_\alpha)_{\alpha \in \Phi} \in U$. Since $U$ is invariant under the action of $GL_1(k)$, we have $(\lambda^\alpha x_\alpha)_{\alpha \in \Phi}$ is also in $U$, for any $\lambda \in k^\times$. On the other hand, as $U$ is a group,
\[ ((\pm \lambda_1^{\alpha} \pm \lambda_2^{\alpha} \pm \cdots \pm \lambda_\ell^{\alpha})x_\alpha)_{\alpha \in \Phi} \in U, \]
for any $\lambda_1, \ldots, \lambda_\ell \in k^\times$. On the other hand, by Lemma 3.4 and the Inverse Function Theorem, the image of $G - G$ has an open neighborhood of the origin. Therefore, thanks to scale invariance of the image, we have: for any $\lambda_1', \ldots, \lambda_\ell' \in k$, one can find $\lambda_1, \mu_1 \in k$ such that
\[ \lambda_1' = \sum_{j=1}^d \lambda_j^{m_j} - \sum_{j=1}^d \mu_j^{m_j}, \]
for any $1 \leq i \leq d$. Now since $p$ is the characteristic of $k$, using (3.1) and (3.2) one can easily finish the argument. □

**Lemma 3.6.** Let $k$ be a local field of characteristic $p$. Let $W$ be a $p$-torsion commutative unipotent $k$-group equipped with a linear $k$-action by $GL_1$; further, assume that all the weights are positive. Let $U$ be a subgroup of $W(k)$, which is invariant under the action of $GL_1(k)$. Then, there exists some $l_0$, depending only on the weights, such that for any integer $l \geq l_0$
\[ U = W'(k^l), \]
where $W'$ is the Zariski-closure of $U$ in $R_{k/k^l}(W)$.

**Proof.** By [CGP10, Prop. B.4.2], we can assume that $W$ is a $k$-vector group equipped with a $k$-linear action of $GL_1$. Applying Lemma 3.5 we can decompose $U$ into subgroups $U_i$ and get a new $GL_1$ action on $\bigoplus_{\alpha \in \Phi_i} W_\alpha$ such that all the new weights are powers of $p$ and $U_i$ is invariant under this new action of $GL_1(K)$. The lemma now follows from Lemma 3.3 □

**Lemma 3.7.** Let $k$ be a local field of characteristic $p$. Let $W$ be a commutative unipotent $k$-group equipped with a $k$-action by $GL_1$ such that $Z_{GL_1}(W) = \{1\}$. Suppose $U$ is a subgroup of $W(k)$, which is invariant under the action of $GL_1(k)$.
Then, there is some $l_0$, depending only on the weights of the action of $\text{GL}_1$ on $\text{Lie}(\mathbb{W})$, such that for any integer $l \geq l_0$

$$\mathcal{U} = \mathbb{W}'(k^q'),$$

where $\mathbb{W}'$ is the Zariski-closure of $\mathcal{U}$ in $\mathcal{R}_{k/k^q'}(\mathbb{W})$.

**Proof.** Since $\mathbb{W}$ is unipotent, it is a torsion group. We now proceed by induction on the exponent of $\mathbb{W}$; if it is $p$, by Lemma 3.6 we are done. Thus, assume that the exponent of $\mathbb{W}$ is $p'$. Let $\mathcal{U}[p] = \{u \in \mathcal{U} | u^p = 1\}$; since $\mathcal{U}$ is commutative, $\mathcal{U}[p]$ is a subgroup of $\mathcal{U}$ which is clearly invariant under the action of $\text{GL}_1(k)$.

Let $\mathcal{V} = \text{Zariski-closure of } \mathcal{U}[p]$ in $\mathbb{W}$; then, $\mathcal{V}$ is a $p$-torsion commutative unipotent group which in view of Lemma 2.3 is defined over $k$. Therefore, by Lemma 3.6 for a large enough power of $p$ (depending only on the weights), which we denote by $q'$, we have $\mathcal{U}[p] = \mathbb{W}'(p)(k^{q'})$, where $\mathbb{W}'(p)$ is the Zariski-closure of $\mathcal{U}[p]$ in $\mathcal{R}_{k/k^{q'}}(\mathcal{V}) \subset \mathcal{R}_{k/k^{q'}}(\mathbb{W})$.

Let $\mathbb{W}''$ be the Zariski-closure of $\mathcal{U}$ in $\mathcal{R}_{k/k^{q'}}(\mathbb{W})$. Note that $\text{GL}_1$ acts on $\mathbb{W}''$ with no trivial weights, and $\mathbb{W}'(p)$ is invariant under this action. Hence both of these groups and their quotient are $k^{q'}$-split unipotent groups. We now consider the following exact sequence of $k^{q'}$-split unipotent groups,

$$1 \to \mathbb{W}'(p) \to \mathbb{W}'' \xrightarrow{\pi} \mathbb{W}'(p)/\mathbb{W}'' \to 1.$$  

Note that $\pi(\mathcal{U})$ is Zariski-dense in $\mathbb{W}'(p)/\mathbb{W}''(p)$ and $p'k^{-1}$-torsion, which implies $\mathbb{W}''(p)/\mathbb{W}'(p)$ is $p^l$-torsion. Hence, by induction hypothesis, there exists $q \geq q'$ which is a large enough power of $p$, depending only on the weights, such that

$$\pi(\mathcal{U}) = \mathbb{W}(k^q),$$

where $\mathbb{W}$ is the Zariski-closure of $\pi(\mathcal{U})$ in $\mathcal{R}_{k^{q'}/k^{q'}}(\mathbb{W}'(p)/\mathbb{W}''(p))$. On the other hand,

$$\mathcal{U}[p] = \mathbb{W}'(p)(k^q) = \mathcal{R}_{k^{q'}/k^q}(\mathbb{W}'(p))(k^q)$$

and, by [Oe83, Cor. A.3.5], $\mathcal{R}_{k^{q'}/k^q}(\mathbb{W}'(p))$ is $k^q$-split unipotent group. Thus $\mathcal{U}[p]$ is Zariski-dense in $\mathcal{R}_{k^{q'}/k^q}(\mathbb{W}'(p))$. By [Oe83, Prop. A.3.8], we also know that the following is exact

$$1 \to \mathcal{R}_{k^{q'}/k^q}(\mathbb{W}'(p)) \to \mathcal{R}_{k^{q'}/k^q}(\mathbb{W}'') \xrightarrow{\pi'} \mathcal{R}_{k^{q'}/k^q}(\mathbb{W}''/\mathbb{W}'(p)) \to 1.$$  

Now let $\mathbb{W}'$ be the Zariski-closure of $\mathcal{U}$ in $\mathcal{R}_{k/k^{q'}}(\mathbb{W}'')$. By the above discussion, it is easy to get the following short exact sequence and show that all of the involved groups are $k^{q'}$-split unipotent groups,

$$1 \to \mathcal{R}_{k^{q'}/k^q}(\mathbb{W}'(p)) \to \mathbb{W}' \xrightarrow{\pi} \mathbb{W} \to 1.$$  

By (3.3) and (3.4) and the fact that these groups are $k^{q'}$-split unipotent groups, we get the following exact sequence,

$$1 \to \mathcal{U}[p] \to \mathbb{W}'(k^q) \xrightarrow{\pi} \pi(\mathcal{U}) \to 1.$$  

So, by (3.5) and $\mathcal{U} \subset \mathbb{W}'(k^q)$, one can easily deduce that $\mathcal{U} = \mathbb{W}'(k^q)$, which finishes the proof. \qed
Proof of Proposition 3.1. We proceed by induction on the nilpotency length of \( W \). If it is commutative, by Lemma 3.7, we are done. Assume \( W \) is of nilpotency length \( c \). Then \([U, U] \subset [W, W](k)\), where \([\cdot, \cdot] \) is the derived subgroup of \( \cdot \). The nilpotency length of \([W, W]\) is \( c - 1 \); hence, by induction hypothesis, for any \( q' \) which is a large enough power of \( p \) (depending only on the weights), we have
\[
[U, U] = \tilde{W}(k^{q'})
\]
where \( \tilde{W} \) is the Zariski-closure of \([U, U]\) in \( R_{k'/k}(\tilde{W}) \). Let \( \mathbb{W}'' \) be the Zariski-closure of \( U \) in \( R_{k'/k'}(\mathbb{W}) \); since \( GL_1 \) acts on \( \mathbb{W}'' \) with no trivial weights and since \( \tilde{W} \) is invariant under this action, both of these groups and the quotient group are \( k' \)-split groups. We consider the following short exact sequence
\[
1 \to \tilde{W} \to \mathbb{W}' \xrightarrow{\pi} \mathbb{W}'' / \tilde{W} \to 1; \tag{3.6}
\]
since \( \pi(U) \) is commutative and Zariski-dense in \( \mathbb{W}' / \tilde{W} \), we get that \( \mathbb{W}' / \tilde{W} \) is commutative. Therefore, by Lemma 3.7 if \( q \geq q' \) is a large enough power of \( p \) (depending only on the weights), we have
\[
\pi(U) = \mathbb{W}(k^q), \tag{3.7}
\]
where \( \mathbb{W} \) is the Zariski-closure of \( \pi(U) \) in \( R_{k'/k}(\mathbb{W}' / \tilde{W}) \). We also have
\[
[U, U] = \mathbb{W}(k^{q'}), \tag{3.8}
\]
By \cite[Prop. A.3.8]{Oe84} and (3.6), we have the exact sequence
\[
1 \to R_{k'/k}(\tilde{W}) \to R_{k'/k}(\mathbb{W}'') \xrightarrow{\pi'} R_{k'/k}(\mathbb{W}' / \tilde{W}) \to 1. \tag{3.9}
\]
Let \( \mathbb{W}' \) be the Zariski-closure of \( U \) in \( R_{k'/k'}(\mathbb{W}'') \). Since \( \tilde{W} \) is a \( k' \)-split unipotent group, by (3.7) and (3.9), we have the following exact sequence
\[
1 \to R_{k'/k}(\tilde{W}) \to \mathbb{W}' \xrightarrow{\pi'} \tilde{W} \to 1
\]
and so, by (3.8), (3.9) and the fact that all the involved groups are \( k^q \)-split unipotent groups, we have
\[
1 \to [U, U] \to R_{k'/k}(\mathbb{W}) \xrightarrow{\pi(U)} 1; \tag{3.10}
\]
this together with \( U \subset \mathbb{W}'(k^{q'}) \) finishes the proof except connectedness.

To see \( \mathbb{W}' \) is connected, note that in view of our assumption that all the weights are positive, there exists some \( r \in k \) so that every element in \( U \) is contracted to the identity by \( r \).

\[\square\]

4. Polynomial like behavior and the basic lemma

In this section we assume \( \mu \) is a probability measure on \( X \) which is invariant under the action of some unipotent \( kT \)-algebraic subgroup of \( G \). We will recall an important construction which shows how polynomial like behavior of the action of unipotent groups on \( X \) can be used to acquire new elements which leave \( \mu \) invariant. Investigation of polynomial orbits in the “intermediate range” in order to take advantage of the polynomial like behavior of diverging orbits dates back to several important works, e.g. Margulis’ celebrated proof of Oppenheim conjecture \cite{Mar86}.
using topological arguments, and Ratner’s seminal work on the proof of measure rigidity conjecture \cite{R90a, R90b, R91}.

4.1. **Construction of quasi-regular maps.** This section follows the construction in \cite{MT94} §5. It is written in a more general setting than what is needed for the proof of Theorem 1.1, namely \( \mu \) is not assumed to be ergodic for the action of the unipotent group \( U \) which is used in the construction. We first recall the definition of a quasi-regular map. Here the definition is given in the case of a local field, which is what we need later, the \( T \)-arithmetic version is a simple modification. It is worth mentioning that we have a simplifying assumption here compared to the situation in \cite{MT94}: our group \( U \) is normalized by an element from class \( A \). This is used in order to define “big balls” in \( U \); in view of this we do not need the construction of the group \( U_0 \) in \cite{MT94}.

**Definition 4.1** (Cf. \cite{MT94}, Definition 5.3). Let \( k \) be a local field.

1. Let \( E \) be a \( k \)-algebraic group, \( U \) a \( k \)-algebraic subgroup of \( E(k) \), and \( M \) a \( k \)-algebraic variety. A \( k \)-rational map \( f : M(k) \to E(k) \) is called \( U \)-quasiregular if the map from \( M(k) \) to \( V \), given by \( x \mapsto \rho(f(x))q \), is \( k \)-regular for every \( k \)-rational representation \( \rho : E \to \text{GL}(V) \), and every point \( q \in V(k) \) such that \( \rho(U)q = q \).

2. Let \( E = E(k) \) and suppose \( U \subset E \) is a \( k \)-split unipotent subgroup. A map \( \phi : U \to E \) is called strongly \( U \)-quasiregular if there exist
   (a) a sequence \( g_n \in E \) such that \( g_n \to e \),
   (b) a sequence \( \{ \alpha_n : U \to U \} \) of \( k \)-regular maps of bounded degree,
   (c) a sequence \( \{ \beta_n : U \to U \} \) of \( k \)-rational maps of bounded degree, and
   (d) a Zariski open dense subset \( X \subset U \),
   such that \( \phi(u) = \lim_{n \to \infty} \alpha_n(u)g_n\beta_n(u) \), and the convergence is uniform on the compact subsets of \( X \).

We note that if \( \phi \) is strongly \( U \)-quasiregular, then it indeed is \( U \)-quasiregular. To see this, let \( \rho : E \to \text{GL}(V) \) be a \( k \)-rational representation, and let \( q \in V \) be a \( U \)-fixed vector. For any \( u \in X \) we have

\[
\rho(\phi(u))q = \lim_{n \to \infty} \rho(\alpha_n(u)g_n)q.
\]

Thanks to the fact that \( U \) is split we can identify \( U \) with an affine space. Then

\[
\psi_n : U \to V \text{ given by } \psi_n(u) = \rho(\alpha_n(u)g_n)q
\]

is a sequence of polynomial maps of bounded degree. Moreover, this family is uniformly bounded on compact sets of \( X \). Therefore, it converges to a polynomial map with coefficients in \( k \). This shows \( \phi \) is \( U \)-quasiregular.

For the rest of this section we assume the following

- \( k \) is a local field,
- \( G \) is the group of \( k \)-points of a \( k \)-group,
- \( U \) is a connected \( k \)-split unipotent subgroup of \( G \),
- there is an element \( s \) from class \( A \) so that \( U \subset W^+_G(s) \), and \( U \) is normalized by \( s \).
Note that in view of these assumptions, [BS68, Prop. 9.13] implies that there a cross section $\mathcal{V}$ for $\mathcal{U}$ in $W_G^+(s)$ which is invariant under conjugation by $s$. Put

$$L := W_G^-(s)Z_G(s)\mathcal{V}.$$ 

Then $L$ is a rational cross section for $\mathcal{U}$ in $G$.

We fix relatively compact neighborhoods $\mathfrak{B}^+$ and $\mathfrak{B}^-$ of $e$ in $W_G^+(s)$, respectively in $W_G^-(s)$, with the property that

$$\mathfrak{B}^+ \subset s\mathfrak{B}^++^{-1}, \quad \text{and} \quad \mathfrak{B}^- \subset s^{-1}\mathfrak{B}^-s.$$

Using these we define a filtration in $W_G^+(s)$ and $W_G^-(s)$ as follows

$$\mathfrak{B}^+_n = s^n\mathfrak{B}^+s^{-n}, \quad \text{and} \quad \mathfrak{B}^-_n = s^{-n}\mathfrak{B}^-s^n;$$

define $\ell^\pm : W^\pm(s) \to \mathbb{Z} \cup \{\pm\infty\}$, by

(i) $\ell^+(x) = j$ iff $x \in \mathfrak{B}^+_j \setminus \mathfrak{B}^+_{j-1}$, and $\ell^+(e) = -\infty$,
(ii) $\ell^-(x) = j$ iff $x \in \mathfrak{B}^-_j \setminus \mathfrak{B}^-_{j-1}$, and $\ell^-(e) = -\infty$.

For any integer $n$, put $U_n = \mathfrak{B}^+_n \cap \mathfrak{B}^-$.

Let $\{g_n\}$ be a sequence in $L\mathcal{U} \setminus N_G(\mathcal{U})$ with $g_n \to e$. Since $L$ is a rational cross-section for $\mathcal{U}$ in $G$, we get rational morphisms

$$\tilde{\phi}_n : \mathcal{U} \to L, \quad \text{and} \quad \omega_n : \mathcal{U} \to \mathcal{U}$$

such that $ug_n = \tilde{\phi}_n(u)\omega_n(u)$ holds for all $u$ in a Zariski open dense subset of $\mathcal{U}$.

Recall that by a theorem of Chevalley, there exists a $k_T$-rational representation $\rho : G \to \text{GL}(\Psi)$ and $q \in \Psi$ such that

$$\mathcal{U} = \{g \in G : \rho(g)q = q\}.$$ 

According to this description we also have

$$\rho(N_G(\mathcal{U}))q = \{z \in \rho(G)q : \rho(\mathcal{U})z = z\}.$$ 

Fix a bounded neighborhood $\mathcal{B}(q)$ of $q$ in the vector space $\Psi$ such that

$$\rho(G)q \cap \mathcal{B}(q) = \overline{\rho(G)q} \cap \mathcal{B}(q),$$

where the closure is taken with respect to the Hausdorff topology of $\Psi$. Recall that $g_n \notin N_G(\mathcal{U})$, thus, in view of (4.3), there is a sequence of integers $\{b(n)\}$ such that $b(n) \to \infty$ and

$$\rho(U_{b(n)+1}g_n)q \subset \mathcal{B}(q), \text{ and } \rho(U_{m}g_n)q \subset \mathcal{B}(q) \text{ for all } m \leq b(n).$$

Define $k$-regular isomorphisms $\tau_n : \mathcal{U} \to \mathcal{U}$ as follows. For every $u \in \mathcal{U}$ put

$$\lambda_n(u) = s^nu^{-1}s^n, \text{ and } \tau_n = \lambda_{b(n)}.$$ 

Given $n \in \mathbb{N}$, we now define the $k$-rational map $\phi_n : \mathcal{U} \to L$ by $\phi_n = \tilde{\phi}_n \circ \tau_n$. Let $\rho_L$ be the restriction to $L$ of the orbit map $g \mapsto \rho(g)q$ and define

$$\phi'_n = \rho_L \circ \phi_n : \mathcal{U} \to \Psi.$$ 

It follows from the definition of $b(n)$ that $\phi'_n(\mathfrak{B}_0) \subset \mathcal{B}(q)$, but $\phi'_n(\mathfrak{B}_1) \subset \mathcal{B}(q)$.

Note that $\phi'_n(u) = \rho(\alpha_n(u)g_n)q$. Hence $\phi'_n : \mathcal{U} \to \Psi$ is a $k$-regular morphism. Also, since

- $\mathcal{U}$ is a connected $k$-group,
• \(\mathcal{U}\) is normalized by \(S\), and
• \(Z_G(S) \cap \mathcal{U} = \{e\}\)

we get from [BS68, Cor. 9.12] that \(\mathcal{U}\) and its Lie algebra are \(S\)-equivariantly isomorphic as \(k\)-varieties. Hence \(\{\phi'_n\}\) is a sequence of equicontinuous polynomials of bounded degree. Therefore, after possibly passing to a subsequence we assume that there exists a \(k\)-regular morphism \(\phi' : \mathcal{U} \to \Psi\) such that

\[
\phi'(u) = \lim_{n \to \infty} \phi'_n(u) \quad \text{for every } u \in \mathcal{U}.
\]

The map \(\phi'\) is non-constant since \(\phi'(B_1)\) in not contained in \(B(q)\); moreover, since \(g_n \to e\) we have \(\phi'_n(e) \to q\), hence \(\phi'(e) = q\).

Let \(\mathcal{M} = \rho(L)q\), since \(L\) is a rational cross section for \(\mathcal{U}\) in \(G\) which contains \(e\), we get that \(\mathcal{M}\) is a Zariski open dense subset of \(\rho(G)q\) and \(v \in \mathcal{M}\). Let now \(\phi : \mathcal{U} \to L\) be the \(k\tau\)-rational morphism defined by

\[
\phi(u) = \rho_L^{-1} \circ \phi'(u);
\]

it follows from the construction that \(\phi(e) = e\) and that \(\phi\) is non-constant.

**Claim.** The map \(\phi\) constructed above is strongly \(\mathcal{U}\)-quasiregular.

To see the claim, first note that by the definition of \(\phi_n\) and in view of (4.7) and (4.8) we have

\[
\phi(u) = \lim_{n \to \infty} \phi_n(u), \quad \text{for all } u \in \phi'^{-1}(\mathcal{M}).
\]

Now since the convergence in (4.7) is uniform on compact subsets and since \(\rho_L^{-1}\) is continuous on compact subsets of \(\mathcal{M}\), we get that the convergence in (4.9) is also uniform on compact subsets of \(\phi'^{-1}(\mathcal{M})\). Recall that \(\tau_n(u)g_n = \phi_n(u)w_n(\tau_n(u))\); hence, for any \(u \in \phi'^{-1}(\mathcal{M})\) we can write

\[
\phi(u) = \lim_{n \to \infty} \tau_n(u)g_n(w_n(\tau_n(u)))^{-1}
\]

which establishes the claim.

### 4.2. Properties of quasi-regular maps and the Basic Lemma

As we mentioned this construction is quite essential to our proof. We will need some properties of the map \(\phi\) constructed above. The proofs of these facts are mutandis mutatis of the proofs in characteristic zero which are given in [MT94], and we will not reproduce the complete proofs in here.

**Proposition 4.2** (Cf. [MT94], §6.1 and §6.3). The map \(\phi\) is a rational map from \(\mathcal{U}\) into \(N_G(\mathcal{U})\). Furthermore, there is no compact subset \(K\) of \(G\) such that \(\text{Im}(\phi) \subset K\mathcal{U}\).

**Proof.** Recall from (4.3) that

\[
N_G(\mathcal{U}) = \{g \in G : \rho(\mathcal{U})\rho(g)q = \rho(g)q\}.
\]

Thus, we need to show that for any \(u_0 \in \mathcal{U}\) we have \(\rho(u_0)\rho(\phi(u))q = \rho(\phi(u))q\) for all \(u \in \phi'^{-1}(\mathcal{M})\); this suffices as \(\phi'^{-1}(\mathcal{M})\) is a Zariski open dense subset of \(\mathcal{U}\). Let \(u \in \phi'^{-1}(\mathcal{M})\) then by (4.10) we have

\[
\phi(u) = \lim_{n \to \infty} \tau_n(u)g_n(w_n(\tau_n(u)))^{-1},
\]

on the other hand \(\rho(u_0\tau_n(u)g_n)q = \rho(\tau_n^{-1}(u_0)u)g_n)q\).
Note now that $\tau_{n}^{-1}(u_{0}) \to e$ as $n \to \infty$, this in view of the above discussion implies that $\phi(u) \in N_{H}(U)$ for all $u \in \phi^{-1}(M)$. The first claim follows.

To see the second assertion, note that $\phi = \rho_{L}^{-1} \circ \phi'$. The claim now follows since $\phi'$ is a non-constant polynomial map, and $\rho_{L}$ is an isomorphism from $L$ onto a Zariski open dense subset of the quasi affine variety $\rho(G)q$.

In the sequel we will construct the quasi-regular map $\phi$ as above using a sequence of elements $g_{n} \to \infty$ with the following property

**Definition 4.3** (Cf. [MT94], Definition 6.6). A sequence $\{g_{n}\}$ is said to satisfy the condition $(\ast)$ with respect to $s$ if there exists a compact subset $K$ of $G$ such that for all $n \in \mathbb{N}$ we have $s^{-b(n)}g_{n}s^{b(n)} \in K$.

This technical condition is used in the proof of the Basic Lemma. It is also essential in the proof of Proposition 5.2.

We also recall the following

**Definition 4.4.** A sequence of measurable non-null sets $A_{n} \subset U$ is called an *averaging net* for the action of $U$ on $(X, \mu)$ if the following analog of the Birkhoff pointwise ergodic theorem holds. For any continuous compactly supported function $f$ on $X$ and for almost all $x \in X$ one has

\[
\lim_{n \to \infty} \frac{1}{\mu(A_{n})} \int_{A_{n}} f(ux) d\theta(u) = \int_{X} f(h) d\mu_{y(x)}(h),
\]

where $\mu_{y(x)}$ denotes the $U$-ergodic component corresponding to $x$.

As we mentioned in the beginning of this section, we will study $U$-orbits of two near by points in the intermediate range. That is: when the maps $\phi_{n}$ are close to their limit. The following states that these pieces of orbits are already equidistributed. The proof is standard, however, it serves as the dynamical counterpart to the above algebraic construction.

**Lemma 4.5** (Cf. [MT94], §7.2). Let $A \subset U$ be relatively compact and non-null. Let $A_{n} = \lambda_{n}(A)$, then $\{A_{n}\}$ is an averaging net for the action of $U$ on $(X, \mu)$.

Our arguments use limiting procedures, such arguments are well adapted to continuous, and not merely measure theoretic, settings. The following definition helps us to address this issue.

**Definition 4.6.** A compact subset $\Omega \subset X$ is said to be a set of uniform convergence relative to $\{A_{n}\}$ if for every $\varepsilon > 0$ and every continuous compactly supported function $f$ on $X$ one can find a positive number $N(\varepsilon, f)$ such that for every $x \in \Omega$ and $n > N(\varepsilon, f)$ one has

\[
\left| \frac{1}{\mu(A_{n})} \int_{A_{n}} f(ux) d\theta(u) - \int_{X} f(h) d\mu_{y(x)}(h) \right| < \varepsilon.
\]

As is proved in [MT94] §7.3], it follows from Egoroff’s Theorem and second countability of the spaces under consideration that: for any $\varepsilon > 0$ one can find a measurable set $\Omega$ with $\mu(\Omega) > 1 - \varepsilon$ which is a set of uniform convergence relative to $\{A_{n} = \lambda_{n}(A)\}$ for every relatively compact non-null subset $A$ of $U$. 
4.3. The following is the main application of the construction of the quasi-regular maps. It provides us with the anticipated “extra invariance property”.

**Basic Lemma** (Cf. [MT94], §7.5, Basic Lemma). Let $\Omega$ be a set of uniform convergence relative to every averaging net $\{A_n = \lambda_n(A)\}$ corresponding to a relatively compact non-null subset $A \subset \mathcal{U}$. Let $\{x_n\}$ be a sequence of points in $\Omega$ with $x_n \to x \in \Omega$. Let $\{g_n\} \subset G \setminus N_G(\mathcal{U})$ be a sequence which satisfies condition $(\ast)$ with respect to $s$; assume further that $g_n x_n \in \Omega$ for every $n$. Suppose $\phi$ is the $\mathcal{U}$-quasiregular map corresponding to $\{g_n\}$ constructed above. Then the ergodic component $\mu_{g(x)}$ is invariant under $\text{Im}(\phi)$.

**Proof.** The proof is the same as in [MT94, Basic Lemma]. Indeed, the analysis simplifies in our situation as $\mathcal{U} = U_0 = \mathcal{U}$ and the map $p$ in [MT94] is the identity map. It is worth mentioning that the technical condition $(\ast)$ is used in the proof as follows: it allows one to write $\omega_n \circ \tau_n$, in the construction of $\phi_n$, as $\tau_n \circ \delta_n$ where $\delta_n : \mathcal{U} \to \mathcal{U}$ is a rational map and locally a diffeomorphism. Furthermore, given $u \in \phi^{-1}(\mathcal{M})$, the sequence $\{\delta_n\}$ will converge uniformly to a diffeomorphism $\delta$ on a neighborhood of $u$. This map is then utilized when applying the ergodic theorem to two “parallel” orbits of $\mathcal{U}$, see also Remark 4.7 below. \hfill \Box

We finish this section with the following remark which will be used in the proof of Theorem 1.1; see Step 4 in the proof.

**Remark 4.7.** The construction above assumed $\{g_n\} \notin N_G(\mathcal{U})$, we, however, make the following observation. Let $\mathcal{U}, \mu$ and $\Omega$ be as in the Basic Lemma, further let us assume that $\mu$ is $\mathcal{U}$ ergodic. Suppose $g \in N_G(\mathcal{U}) \cap Z_G(s)W^+_G(s)$ is so that $gx \in \Omega$ for some $x \in \Omega$. Then $\mu$ is invariant by $g$.

To see this, put $A = \mathfrak{B}_0$. For all $n \geq 0$ and any continuous compactly supported function $f$ we have

\[
\frac{1}{\theta(A_n)} \int_{A_n} f(ugx)d\theta(u) = \frac{1}{\theta(A)} \int_A f(\lambda_n(u)gx)d\theta(u) \\
= \frac{1}{\theta(A)} \int_A f(gg^{-1}\lambda_n(u)gx)d\theta(u) \\
= \frac{1}{\theta(A)} \int_A f(g\lambda_n(g^{-1}ug_n)x)d\theta(u) \\
= \frac{1}{\theta(B(n))} \int_{B(n)} f(g\lambda_n(u)x)d\theta(u),
\]

where $g_n = \lambda_n^{-1}(g)$ and $B(n) = g_n^{-1}Ag_n$. In the last equality we used the fact that the Jacobian of the conjugation by $g_n$ is constant.

Now let $g_0 \in N_G(\mathcal{U})$ be so that $g_n \to g_0$ as $n \to \infty$. Put $B = g_0^{-1}Ag_0$, then $\theta(B(n) \Delta B) \to 0$ as $n \to \infty$. Hence, for any $\varepsilon > 0$ and all large enough $n$ have

\[
\left| \frac{1}{\theta(B(n))} \int_{B(n)} f(g\lambda_n(u)x)d\theta(u) - \frac{1}{\theta(B)} \int_B f(g\lambda_n(u)x)d\theta(u) \right| \leq \varepsilon
\]

\footnote{Indeed one does not need the assumption $g \in Z_G(s)W^+_G(s)$ for the conclusion to hold. This however suffices for our application, moreover, the proof under this assumption has some similarities with the change of variable which is used in the proof of the Basic Lemma in [MT94] which we wish to highlight.}
On the other hand since $x, gx \in \Omega$, for all large enough $n$ we have
\[
\left| \frac{1}{|A_n|} \int_{A_n} f(ugx)d\theta(u) - \int_X f(h)d\mu(h) \right| < \varepsilon \quad \text{and} \quad \left| \frac{1}{|B_n|} \int_{B_n} f(gux)d\theta(u) - \int_X f(h)d\mu(h) \right| < \varepsilon.
\]
Putting all these together we get $|\mu(f) - g\mu(f)| \leq 3\varepsilon$. This implies the claim if we let $\varepsilon$ tend to 0.

5. **Proof of Theorem 1.1**

Let us recall the set up from §1 We fixed a $k_{T}$-algebraic group $G$ a closed subfield $k' \subset k_w$ together with a $k_w$-homomorphism from $H$ into $G_w$. Therefore, if we replace $G_w$ by $G_w' = R_{k_s/k'}(G_w')$, we get a $k'$-algebraic group, $G'$, such that $G'(k') = G(k_w)$. Furthermore, it follows from the universal property of Weil’s restriction of scalars that $H$ is a $k'$-algebraic subgroup of $G'$. Therefore, we may and we will assume that $k' = k_w$, and $G_w = G_w'$. To simplify the notation, we will denote $k = k_w$ for the rest of this section.

We also fixed a non central homomorphism $\lambda : G_m \to \mathbb{H}$. Using this we defined a one dimensional $k$-split torus $S \subset H$, a one dimensional $k$-split tours $S \subset G$, and an element $s \in S$ from class $A$.

Let $U \subset W_{G}^{+}(s)$ be a closed, with respect to the Hausdorff topology, subgroup normalized by $S$. Then, by Proposition 3.4 we have the following. There exists some $q = p^n$, depending on the action of $S$ on $W_{G}^{+}(s)$, such that $U$ is the group of $k^{\sigma}$-points of a connected $k'$-algebraic unipotent subgroup of $R_{k/k'}(W_{G}^{+}(s))$. Replacing $G_w$ with $R_{k/k'}(G_w)$, which we continue to denote by $G_w$, we have $U$ is an algebraic subgroup of $G_w$.

Note that we have now replaced $H$ with $R_{k/k'}(H)$ which is of the form we discussed in the introduction. We fix a $k'$-homomorphism $\lambda' : G_m \to R_{k/k'}(H)$ so that $\lambda'(\varpi^q)$ is power of $s$. Finally replace $S$ by $\lambda'(G_m)$, $S$ by $R_{k/k'}(\lambda'(G_m))(k')$ and $s$ by the image of $\lambda'(\varpi^q)$. After these observations we may and will replace $k$ by $k^q$ in $k_T$, and abusing the notation, we will still denote this by $k_T$. In particular, now $U$ is defined over $k$ and hence over $k_T$ and is normalized by the $k$-split one dimensional $k$-torus $s$.

5.1. **The subgroup $U$**. Recall that $\mu$ is a probability measure on $X$ which is $SU$-invariant and $U$-ergodic. Define

(5.1) \[
U \subset W_{G}^{+}(s) \quad \text{to be the maximal subgroup leaving $\mu$ invariant.}
\]

Note that $U \subset U$. The group $U$ is a closed, in Hausdorff topology, subgroup of $W_{G}^{+}(s)$ which is normalized by $S$ since $\mu$ is $S$-invariant and $W_{G}^{+}(s)$ is normalized by $S$.

The above discussion thus implies that, after a certain restriction of scalars which possibly replaces $k$ with an inseparable subfield, we may assume

- the group $U$ is $k$-points of a connected $k$-split unipotent subgroup of $W_{G}^{+}(s)$,
- the group $S$ is a $k$-split one dimensional $k$-torus and $s \in S$. 

Following [MT94], we define

\[ F(s) = \{ g \in G : U g \subset W_G^-(s)Z_G(s)U \} , \]

here the closure is the Zariski closure. Since \( W_G^-(s)Z_G(s) \) is a subgroup of \( G \) the above can be written as

\[ \{ g \in G : \overline{W_G^-(s)Z_G(s)U} g \subset \overline{W_G^-(s)Z_G(s)U} \} . \]

Thus the inclusion in (5.3) may be replaced by equality. This implies

\[ F(s) \] is a \( k \)-closed subgroup of \( G \).

Note that \( S \subset F(s) \) and \( F(s) \cap W_G^+(s) = U \).

Put \( U^- = F(s) \cap W_G^-(s) \), this a \( k \)-closed subgroup of \( W_G^-(s) \) which is normalized by \( s \).

**Lemma 5.1.** The group \( U^- \) is the \( k \)-points of a connected unipotent subgroup of \( W_G^-(s) \). Moreover, there is an \( s \)-invariant cross-section \( V^- \) for \( U^- \) in \( W_G^-(s) \).

**Proof.** The second claim follows from the first claim and [BS68, Prop. 9.13].

To see the first claim, note that by Lemma 2.3 there is a smooth group scheme \( B \) defined over \( k \) so that \( B(k) = U^- \). Hence, \( B \subset W_G^- \) is a unipotent group which is normalized by \( s \). Since \( s \) contracts every element of \( B \) to the identity we get \( B \) is connected as was claimed.

\[ \square \]

Similarly we fix an \( s \)-invariant cross-section \( V \) for \( U \) in \( W_G^+(s) \). We will use the notation and construction in [4] with this \( U \) and \( s \).

5.2. **Structure of \( \mu \) along contacting leaves of \( s \).** In this section we will use the maximality of \( U \) and the Basic Lemma to show that \( \mu \) has a rather special structure along \( W_G^- \). The main result is Proposition 5.3. We first need some more notation. Put

\[ D = W_G^-(s)Z_G(s)W_G^+(s) = U^-V^-Z_G(s)VU. \]

Then \( D \) is a Zariski open dense subset of \( G \) containing \( e \), see [2.1]. Moreover, for any \( g \in D \) we have a unique decomposition

\[ g = w^-gz(g)w^+(g) = u^-g v^-g z(g)v(g)u(g) \]

where \( u^- \in U^- \), \( v^- \in V^- \), \( z \in Z_G(s) \), \( u \in U \), \( v \in V \), \( w^- = u^-g v^-g \), and \( w^+ = v(g)u(g) \).

Note that for every \( w^\pm \in W_G^\pm(s) \) we have

\[ \ell^\pm(s^m w^\pm s^{-m}) = \ell^\pm(w^\pm(g)) \pm m. \]

We need the following

**Proposition 5.2** (Cf. [MT94], Proposition 6.7). Suppose \( \{ g_n \} \) is a sequence converging to \( e \), and let \( s \) and \( U \) be as above. Suppose one of the following holds

(i) the sequence \( \ell^-(v^-g_n) - \ell^-(u^-g_n) \) is bounded from below, or
(ii) \( \{ g_n \} \subset Z_G(s)W_G^+(s) \setminus N_G(U) \).
Then \( \{ g_n \} \) satisfies condition (\( * \)). Furthermore, if we let \( \phi \) be the quasi-regular map constructed using \( \mathcal{U} \) and this sequence \( \{ g_n \} \), then \( \text{Im}(\phi) \subset W^+_G(s) \).

**Proof.** The fact that the conclusion holds under condition (i) is proved in [MT94 Prop. 6.7]. We show (ii) also implies the conclusion; under this assumption we have \( s^{-b(n)}g_n s^{b(n)} \rightarrow e \), hence, \( \{ g_n \} \) satisfies condition \( (\ast) \). We now use an argument similar to [MT94 Prop. 6.7] to show \( \text{Im}(\phi) \subset W^+_G(s) \) when condition (ii) above holds. By (4.9) we have

\[
\phi(u) = \lim_{n \to \infty} \phi_n(u) = \lim_{n \to \infty} \tau_n(u)g_n \omega_n(\tau_n(u))^{-1} \text{ for all } u \in \phi^{-1}(M).
\]

It follows from the choice of \( b(n) \) that

\[
\{ s^{-b(n)} \cdot s^{b(n)} \} \text{ are bounded in } \mathcal{U} \text{ for } \bullet = \tau_n(u) \text{ and } \omega_n(\tau_n(u))^{-1}.
\]

In view of this and condition \( (\ast) \) we get the following. After possibly passing to a subsequence, we have

\[
\lim_{n \to \infty} s^{-b(n)} \phi_n(u) s^{b(n)} = \lim_{n \to \infty} s^{-b(n)} \tau_n(u)g_n \omega_n(\tau_n(u))^{-1} s^{b(n)} \in \mathcal{U},
\]

which implies \( \phi(u) \in W^+_G(s) \) for all \( u \in \phi^{-1}(M) \). This together with the fact that \( \phi^{-1}(M) \) is Zariski dense in \( \mathcal{U} \) finishes the proof. \( \square \)

The following is an important consequence of the above proposition and the construction of quasi-regular maps in \[44\]. It describes the local structure of the set of uniform convergence. Our formulation here is taken from [MT94], let us remark that obtaining such description is also essential in Ratner’s proof in \[90b\].

**Proposition 5.3** (Cf. [MT94], Proposition 8.3). For every \( \varepsilon > 0 \) there exists a compact subset \( \Omega_\varepsilon \) of \( X \) with \( \mu(\Omega_\varepsilon) > 1 - \varepsilon \) such that if \( \{ g_n \} \subset G \setminus NC(U^+(s)) \) is a sequence so that \( g_n \rightarrow e \) and

\[ g_n \Omega_\varepsilon \cap \Omega_\varepsilon \neq \emptyset \text{ for every } n, \]

then the sequence \( \{ \ell^- (v^-(g_n)) - \ell^- (u^-(g_n)) \} \) tends to \( -\infty \).

**Proof.** First note that \( U \subset \mathcal{U} \), therefore, \( \mu \) is \( \mathcal{U} \)-ergodic and invariant. Let \( \varepsilon > 0 \) be given and let \( \Omega_\varepsilon \) be a set of uniform convergence for the action of \( \mathcal{U} \), in the sense of Definition 4.6, with \( \mu(\Omega_\varepsilon) > 1 - \varepsilon \). We will show that \( \Omega_\varepsilon \) satisfies the conclusion of the proposition.

Assume the contrary, and let \( \{ g_n \} \) be a sequence for which the conclusion of the proposition fails for \( \Omega_\varepsilon \). Passing to a subsequence we may assume the sequence \( \{ \ell^- (v^-(g_n)) - \ell^- (u^-(g_n)) \} \) is bounded from below. Therefore, Proposition 5.2 guarantees that \( \{ g_n \} \) satisfies condition \( (\ast) \). Now construct the quasi-regular map \( \phi \) corresponding to \( \{ g_n \} \) as in [44]. It follows from the Basic Lemma that \( \mu \) is invariant under \( \text{Im}(\phi) \). Furthermore, by Proposition 5.2 we have \( \text{Im}(\phi) \subset W^+_G(s) \), and in view of Proposition 4.7 the image of \( \phi \) is not contained in \( \mathcal{KU} \) for any bounded subset \( \mathcal{K} \subset G \). Therefore, \( \mu \) is invariant under \( (\mathcal{U}, \text{Im}(\phi)) \) which is contained in \( W^+_G(s) \) and strictly contains \( \mathcal{U} \). This contradicts the maximality of \( \mathcal{U} \). \( \square \)

We will use this proposition in the following form.
**Corollary 5.4** (Cf. [MT94], Corollary 8.4). There exists a subset $\Omega \subset X$ with $\mu(\Omega) = 1$ such that

$$W^{-}_G(s)x \cap \Omega \subset U^- x$$

for every $x \in \Omega$.

**Proof.** The proof follows the same lines as the proof of [MT94] Cor. 8.4 using Proposition 5.3. We recall the proof here for the convenience of the reader. First, let us note that by Mautner’s phenomena every $s$-ergodic component of $\mu$ is $U$-invariant, thus, $\mu$ is $s$-ergodic. For any $\varepsilon > 0$ let $\Omega_\varepsilon$ be as in Proposition 5.3. Let $\Omega'_\varepsilon \subset \Omega_\varepsilon$ be a compact subset with $\mu(\Omega'_\varepsilon) > 1 - 2\varepsilon$ so that the Birkhoff ergodic theorem for the action of $s$ and $\chi_{\Omega_\varepsilon}$ holds for every $x \in \Omega'_\varepsilon$. Suppose $x$ and $y = w^- x$ are in $\Omega'_\varepsilon$, and assume $w^- \notin U^-$. Let $n_i \to \infty$ be a subsequence so that both $s^{n_i} x \in \Omega_\varepsilon$ and $s^{n_i} y \in \Omega'_\varepsilon$, such sequence exists by Birkhoff ergodic theorem. Put $x_i = s^{n_i} x$ and

$$y_i = s^{n_i} y = s^{n_i} ws^{-n_i} s^{n_i} x = w_i x_i$$

where $w_i = s^{n_i} ws^{-n_i}$.

Our assumption on $w$ and (5.6) imply that $\{\ell^-(v^-(w_i)) - \ell^-(u^-(w_i))\}$ is bounded from below which contradicts Proposition 5.3. The corollary now follows by applying the above to a sequence $\varepsilon_n \to 0$ and by putting $\Omega = \cup_{n} \Omega'_{\varepsilon_n}$. \qed

### 5.3. A lemma on finite dimensional representations.

We need certain properties of the subgroup $F(s)$, which was defined in (5.2). These will be used when we apply Theorem 5.8 in the proof of Theorem 1.1. The main property needed is Lemma 5.7 below, which is a consequence of Lemma 5.5. It is worth mentioning that the latter is closely related to the notion of an *epimorphic group* which was introduced by A. Borel.

Retain the notation from (5.1). Put $H^+ = \langle W^+_G(s), W^-_G(s) \rangle$. Since $\mathbb{H}'$ is $k''$-almost simple we have

$$H^+ = i(\mathcal{R}_{k''/k}(\mathbb{H}))(k)$$

where $\mathbb{H}' \to \mathbb{H}'$ denotes the simply connected cover of $\mathbb{H}'$, and $i$ is $\iota$ precomposed with the covering map, see [Mar90] Prop. 1.5.4 and Thm. 2.3.1].

**Lemma 5.5** (Cf. [Sh95], Lemma 5.2). Let $(\rho, \Phi)$ be a finite dimensional representation of $G_w$ defined over $k$ and let $\|\|$ denote a norm on $\Phi$. Let $q \in \Phi$.

1. If $\{U, s\} \subset \{g \in G_w : \rho(g)q \in k \cdot q\}$, then $\|\rho(s^n)q\| \geq \|q\|$ for all $n \geq 1$.
2. If $\rho(s)q = q$ and $\rho(U)q = q$, then $\rho(H^+)q = q$.

**Proof.** First note that since $U$ is a unipotent subgroup of $G$ our assumption in (1) implies that $\rho(U)q = q$, and that $\rho(s)q = \chi(s)q$ for some $k$-character $\chi$.

Assume the contrary to (1), then since $s$ acts by a character, we get that

$$\lim_{n \to \infty} \rho(s^n)q = 0.$$  

Let $\Phi^-$ denote the subspace of $\Phi$ corresponding to the negative weights of the action of $\rho(s)$ on $\Phi$. In this notation, (5.7) is to say $q \in \Phi^-$. We claim (5.7) implies the following

$$\rho(H^+)q \subset \Phi^-.$$
Let us assume (5.8) and conclude the proof first. Indeed (5.8) implies that
\[ \Psi := k\text{-linear span of } \{ \rho(H^+)q \} \subset \Phi^- . \]

Let \( q : H^+ \to GL(\Psi) \) be the corresponding representation. Now by Lemma 5.6 there is some \( n_0 \) so that \( s^{n_0} \in H^+ \), hence
\[ \det(q(s^{n_0})) \neq 1. \]

This contradicts the fact that \( H^+ \) is a generated by \( k\)-unipotent subgroups and proves the lemma.

We now turn to the proof of (5.8). Recall that \( q \) is fixed by \( U \), therefore, (5.8) follows if we show that
\[ \rho(W_H^{-}(s)) \subset \Phi^- . \]

Let \( w \in W_H^{-}(s) \) be arbitrary. Then \( s^n w s^{-n} \to e \) as \( n \to \infty \). Using this and (5.7) we get that
\[ \lim_{n \to \infty} \rho(s^n) \rho(w) q = \lim_{n \to \infty} \rho(s^n w s^{-n}) \rho(s^n) q = 0 . \]

Hence, \( \rho(w) q \in \Phi^- \) for all \( w \in W_H^{-}(s) \) as we wanted to show.

We now prove (2); the proof is similar to the above. Decompose \( \Phi \) according to the weights of our fixed element \( s \in S \). Hence
\[ \Phi = \Phi^- + \Phi^0 + \Phi^+ . \]

We claim that
\[ (5.9) \quad \rho(W_H^{-}(s)Z_H(s))q \subset \Phi^- + \Phi^0 . \]

To see this, let \( wz \in W_H^{-}(s)Z_H(s) \) be arbitrary. Then \( s^n w s^{-n} \to e \) as \( n \to \infty \) and \( sz = zs \). Since \( \rho(s)q = q \) we get that
\[ \lim_{n \to \infty} \rho(s^n) \rho(wz) q = \lim_{n \to \infty} \rho(s^n w s^{-n}) \rho(z) \rho(s^n) q = \rho(z) q . \]

Hence, \( \rho(wz) q \in \Phi^- + \Phi^0 \) for all \( wz \in W_H^{-}(s)Z_H(s) \) as we wanted to show.

Our assumption that \( \rho(U)q = q \) together with (5.9) now implies that
\[ \rho(W_H^{-}(s)Z_H(s)W_H^{-}(s))q \subset \Phi^- + \Phi^0 . \]

This is to say \( \rho \circ \iota(\mathbb{W}_H(s)(k')Z_H(s)(k')\mathbb{W}_H^{-}(s)(k'))q \subset \Phi^- + \Phi^0 . \) In view of (2.1) we thus get
\[ \rho(H^+)q \subset \rho(H)q = \rho(\iota(\mathbb{H}(k_w)))q \subset \Phi^- + \Phi^0 . \]

As above, define \( \Psi \) to be the \( k\)-span of \( \{ \rho(H)q \} \); note that \( \Psi \subset \Phi^- + \Phi^0 \). Let \( (q, \Psi) \) denote the corresponding representation of \( H \) on \( \Psi \). Let \( n_0 \) be so that \( s^{n_0} \in H^+ \).

Since \( H^+ \) is generated by \( k_w\)-unipotent subgroups we get that
\[ \det(q(s^{n_0})) = 1 , \]

which implies that \( \Psi \subset \Phi^0 \).

Let now \( p \in \Psi \) be any vector. For any compact subset \( B \subset U \) there is a compact subset \( B' \subset \Psi \) so that \( q(B)p \subset B' \). Since \( \Psi \subset \Phi^0 \), we get
\[ (5.10) \quad q(s^{n_0} Bs^{-n_0})p = q(s^n)q(B)p \subset B' . \]
Letting \( n \to \infty \) we get from (5.10) that \( \rho(U)p \subset B' \) is contained in a compact subset of \( \Psi \). Note, however, that \( U \) is a \( k_w \)-split \( k_w \)-unipotent subgroup, therefore

\[
\rho(U)p = p.
\]

Hence, \( U \) is in the kernel of \( \rho \). Since the kernel is a normal subgroup of \( H^+ \) we get from Lemma 5.6 that \( H^+ = \ker(\rho) \), which implies (2). \( \square \)

**Lemma 5.6.** The only normal subgroup of \( H^+ \) which contains \( U \) is \( H^+ \). Moreover, \( H/H^+ \) is a compact and torsion group.

**Proof.** Note that \( \mathbb{H}(k') = \mathbb{H}'(k''') \) and \( s \in \iota(\mathbb{H}(k')) \). Therefore, \( H^+ = \iota((\mathbb{H}'(k'''))^+) \).

Since \( \mathbb{H}' \) is a \( k'' \)-almost simple group, the lemma holds for \( (\mathbb{H}'(k'''))^+ \), see [Mar90b, Prop. 1.5.4, Thm. 1.5.6 and Thm. 2.3.1]. Hence it holds for \( H^+ \). \( \square \)

Let \( F \) denote the Zariski closure of the \( k \)-closed group \( F(s) \). By Lemma 2.3 we have \( F \) is a \( k \)-algebraic subgroup of \( G \) and \( F(s) = F(k) \). It is worth mentioning that \( F \) is not necessarily connected we let \( F = F^c(k) \). Using the adjoint action of \( s \) we have

\[
\text{Lie}(F) = \text{Lie}(U^-) \oplus \text{Lie}(F \cap Z_G(s)) \oplus \text{Lie}(U) \subset \text{Lie}(G).
\]

Recall from the introduction that the product map

\[
(5.11) \quad U^- \times (F \cap Z_G(s)) \cap U \rightarrow F
\]

is a diffeomorphism onto a Zariski open dense subset which contains the identity.

Let us fix a norm \( \| \cdot \| \) on \( \text{Lie}(G) \). Put \( \Phi = \wedge^{\dim F} \text{Lie}(G) \), \( \rho = \wedge^{\dim F} \text{Ad} \) and let \( q \in k \cdot \wedge^{\dim F} \text{Lie}(F) \) be a nonzero vector. Then

\[
(5.12) \quad F(s) \subset \{ g \in G : \rho(g)q \in k \cdot q \},
\]

moreover, since \( U \) is a unipotent subgroup \( \rho(U)q = q \). Recall now that \( U \subset U \), therefore,

\[
(5.13) \quad \rho(U)q = q.
\]

**Lemma 5.7.** We have

\[
(5.14) \quad \alpha(s,U) \geq \alpha(s^{-1},U^-)
\]

where the function \( \alpha \) denotes the modulus of the conjugation action, see § 2.1

**Proof.** We first note that in view of relations between the Haar measure and algebraic form of top degree, see [Bour, 10.1.6] and [Oe84, Thm. 2.4], this is equivalent to the fact that

\[
(5.15) \quad |\det(\text{Ad}(s))|_{\text{Lie}(U)}|_w \geq |\det(\text{Ad}(s^{-1}))|_{\text{Lie}(U^-)}|_w.
\]

In view of the definition of \( \rho \) and \( q \), (5.15) follows if we show \( \|\rho(s)q\| \geq \|q\| \). The latter holds thanks to Lemma 5.5(1) in view of (5.12) and (5.13). \( \square \)
5.4. Entropy argument and the conclusion of the proof. The following theorem is proved in \[MT94\] and serves as one of the main ingredients in the proof of the measure classification theorem in \[MT94\]. Given an element \(s\) from class \(A\) which acts ergodically on the measure space \((X, \sigma)\) we let \(h_\mu(s)\) denote the measure theoretic entropy of \(s\).

**Theorem 5.8** (Cf. \[MT94\], Theorem 9.7). Assume \(s\) is an element from class \(A\) which acts ergodically on a measure space \((X, \sigma)\). Let \(V\) be an algebraic subgroup of \(W_G^-\) which is normalized by \(s\) and put \(\alpha = \alpha(s^{-1}, V)\).

1. If \(\sigma\) is \(V\)-invariant, then \(h_\mu(s) \geq \log \alpha\).
2. Assume that there exists a subset \(\Omega \subset X\) with \(\sigma(\Omega) = 1\) such that for every \(x \in \Omega\) we have \(W^- G(x) \cap \Omega \subset Vx\). Then \(h_\mu(s) \leq \log(\alpha)\) and the equality holds if and only if \(\sigma\) is \(V\)-invariant.

**Proof of Theorem 1.1.** Let \(\mu\) be as in the statement of Theorem 1.1. First note that \(\mu\) is \(s\)-ergodic. Indeed by Mautner phenomena any \(s\)-ergodic component of \(\mu\) is \(U\)-invariant. Therefore, \(s\)-ergodicity follows from the fact that \(\mu\) is \(U\)-ergodic.

Let \(U\) be as in \[5.1\], i.e. \(U\) is the maximal subgroup of \(W_G^\pm\) which leaves \(\mu\) invariant. We now complete the proof in some steps.

**Step 1.** \(\mu\) is invariant under \(U^\pm\).

By Corollary 5.3, there exists a full measure subset \(\Omega \subset X\) such that
\[
W^- G(x) \cap \Omega \subset \Omega U^\pm x \quad \text{for every} \quad x \in \Omega.
\]
Recall that \(\mu\) is \(s\)-ergodic. Applying Theorem 5.8 we get
\[
\log \alpha(s, U) \leq h_\mu(s) = h_\mu(s^{-1}) \leq \log \alpha(s^{-1}, U^-).
\]
Note, however, that by Lemma 5.7 we have
\[
\alpha(s, F(s)) = \alpha(s, U^-) \alpha(s, U) \geq 1,
\]
moreover, \(\alpha(s^{-1}, U^-) = \alpha(s, U^-)^{-1}\). Therefore, equality must hold in \[5.16\]. Now another application of Theorem 5.8(ii) implies that \(\mu\) is invariant under \(U^\pm\).

**Step 2.** Reduction to Zariski dense measures.

We now apply Lemma 2.5 with \(B = \langle U^\pm, S, U\rangle\) and \(M = G\). Hence we get a connected \(kT\)-subgroup \(G \subset \mathbb{G}\), and a point \(gT = x \in X\) such that \(B \subset G^\prime\) and \(\mu(G^\prime x) = 1\) where \(G^\prime := G(kT)\). Moreover, for every proper \(kT\)-closed subset \(D \subset G^\prime\) we have \(\mu(\pi(D)) = 0\). and \(G^\prime \cap gT^{-1}\) is Zariski dense in \(G^\prime\).

Abusing the notation we let \(V\) (resp. \(V^-\)) denote an \(S\)-invariant cross section for \(U\) (resp. \(U^-\)) in \(W^+_G\) (resp. \(W^- G\)).

**Step 3.** \(\mu\) is invariant under \(W^- G\).

We will show \(U^- = W^- G(s)\) which implies the claim in view of Step 1. Assume the contrary. Then \(V^- \neq \{e\}\). The definition of \(U^-\), see \[5.2\], implies that
\[
V^- Z G^\prime(s) W^+_G(s) \not\subset N G^\prime(U).
\]
In particular,
\[
D' := (Z G^\prime(s) W^+_G(s) \cup N G^\prime(U)) \cap V^- Z G^\prime(s) W^+_G(s)
\]
is a proper $k_T$-closed subset of $V^c \subset (s) W^+_{G'}(s)$.

This together with Step 1 and Step 2 implies that conditions in Lemma 2.6 are satisfied with $M = G'$, $B = U^c$, $L = V^c \subset (s) W^+_{G'}(s)$, and $D = D'$. Therefore, we get from the conclusion of that lemma the following. If $0 < \varepsilon < 1$ is small enough and $\Omega_\varepsilon$ is a measurable set with $\mu(\Omega_\varepsilon) > 1 - \varepsilon$, then there exists a sequence $\{g_n\}$ converging to $e$ such that

\[
\{g_n\} \subset V^c \subset (s) W^+_{G'}(s) \setminus (Z_{G'}(s) W^+_{G'}(s) \cup N_{G'}(U)),
\]

and $g_n \Omega_\varepsilon \cap \Omega_\varepsilon \neq \emptyset$ for all $n$.

In particular, we have $\ell^c(\varepsilon(g_n)) > -\infty$ and $\ell^c(\varepsilon(g_n)) = -\infty$. This contradicts Proposition 5.3 and shows that $U^c = W^+_{G'}(s)$ as was claimed.

**Step 4.** $\mu$ is invariant under $W^+_{G'}(s)$.

This will follow if we show $U = W^+_{G'}(s)$. Assume the contrary, and let $\Omega_\varepsilon$ be as in Step 3. Then $Z_{G'}(s) U$ is a proper subvariety of $Z_{G'}(s) W^+_{G'}(s)$.

Apply Lemma 2.6 with $M = G'$, $B = W^+_{G'}(s)$, $L = Z_{G'}(s) W^+_{G'}(s)$ and $D = Z_{G'}(s) U$. Therefore, we find

\[
\{g_n\} \subset Z_{G'}(s) W^+_{G'}(s) \setminus Z_{G'}(s) U
\]

so that $g_n \Omega_\varepsilon \cap \Omega_\varepsilon \neq \emptyset$, and $g_n \to e$. There are two cases to consider.

**Case 1.** Suppose there is a subsequence $\{g_{n_i}\}$ such that $g_{n_i} \notin N_{G'}(U)$ for all $i$. Abusing the notation we denote this subsequence by $\{g_n\}$. Construct the map $\phi$ using this sequence $\{g_n\}$. Then $\mu$ is invariant by $\langle U, \text{Im}(\phi) \rangle$. On the other hand by Proposition 5.2(ii) we have $\text{Im}(\phi) \subset W^+_{G'}(s) \setminus U$. This contradicts the maximality of $U$ and finishes the proof.

**Case 2.** There exists some $n_0$ so that $g_n \in N_{G'}(U)$ for all $n \geq n_0$. Taking $n \geq n_0$ we assume $g_n \in N_{G'}(U)$. Now by Remark 4.7 $\mu$ is invariant under $g_n$ for all $n$. Write

\[
g_n = z(g_n)v(g_n)u(g_n) \in Z_{G'}(s) \Delta U.
\]

Since $\mu$ is $U$-invariant, the above implies that $\mu$ is invariant under $z(g_n)v(g_n)$. Moreover, in view of the choice of $g_n$ we have $v(g_n) \notin e$. Recall also that $\mu$ is $s$-invariant. Therefore, $\mu$ is invariant under

\[
s^\ell z(g_n)v(g_n)s^{-\ell} = z(g_n)s^\ell v(g_n)s^{-\ell}.
\]

for all $n \in \mathbb{Z}$. For each $n$ choose $\ell_n \in \mathbb{Z}$ so that

\[
v_n = s^{\ell_n} v(g_n)s^{-\ell_n} \in (V \cap \mathfrak{H}_0) \setminus \mathfrak{H}_{-1} ;
\]

such $\ell_n$ exists since $v(g_n) \neq e$, and the cross section $V$ is $S$-invariant. Now passing to a subsequence we get $z(g_n)v_n \to v \in V$ and $v \neq e$. Therefore, $\mu$ is invariant by $v$. This again contradicts maximality of $U$.

**Step 5.** Conclusion of the proof.

So far we have proved that $\mu(G' x) = 1$ and $\mu$ is invariant and ergodic under $G'' := \langle W^+_{G'}(s), W^+_{G'}(s) \rangle$. Hence $\mu$ is a probability measure on $G'/G' \cap q G^{-1}$ which is invariant and ergodic under $G''$. Note now that in view of the definitions (2.2) and (2.4), Lemma 2.1 implies that $G''$ is a normal and unimodular subgroup of $G$. Theorem 1.1 now follows from this and Lemma 2.1. 

\[\square\]
We now give a refinement of Theorem 1.1.

**Theorem 5.9.** Let the notation and the assumptions be as in Theorem 1.1. Then, there exist

1. \( l_T = \prod_{v \in T} l_v \subset k_T \) where \( l_v = k_v \) if \( v \neq w \), and \( l_w = (k')^q \) for some \( q = p^n \) where \( q \) depends on the weights of the action of \( S \).

2. a connected \( l_T \)-subgroup \( F \) of \( \mathcal{R}_{k_T/l_T}(G) \) so that \( F(l_T) \cap \Gamma \) is Zariski dense in \( F \),

3. a point \( x = g_0 \Gamma \in X \),

such that \( \mu \) is the \( \Sigma \)-invariant probability Haar measure on the closed orbit \( \Sigma x \) with

\[
\Sigma = g_0 F^+(\lambda)(F(l_T) \cap \Gamma)g_0^{-1}
\]

where

- the closure is with respect to the Hausdorff topology, and
- \( F^+(\lambda) \) is defined by (2.2) for a non central \( l_T \)-homomorphism \( \lambda : G_m \to \mathbb{F} \).

**Proof.** Indeed the above assertions are proved in the course of the proof of Theorem 1.1. We give a more detailed discussion here for the sake of completeness.

In view of the discussion in the beginning of §5 we have the following.

(a) There is some \( l_T \subset k_T \) as in (1) so that the group \( \mathcal{U} \) which is defined in (5.1) is an \( l_T \)-split unipotent subgroup of \( \mathcal{R}_{k_T/l_T}(G) \).

We thus replaced \( G \) with \( \mathcal{R}_{k_T/l_T}(G) \) and have \( SU \subset G \), see [5.1] By Step 2 in the proof of Theorem 1.1 we have the following.

(b) There is a connected \( l_T \)-subgroup \( E \) of minimal dimension such that \( SU \subset E := E(l_T) \) and a point \( x = g_0 \Gamma \in X \) with the following properties. \( \mu \) is a probability measure on \( E/E \cap g_0 \Gamma g_0^{-1} \), moreover, \( E \cap g_0 \Gamma g_0^{-1} \) is Zariski dense in \( E \).

By Step 5 in the proof of Theorem 1.1 we have the following.

(c) \( \mu \) is the \( \Sigma \)-ergodic invariant measure on the closed orbit the closed orbit \( \Sigma x \), where

\[
\Sigma = E'(E \cap g_0 \Gamma g_0^{-1}),
\]

with \( E' := \langle W^+(s), W^-(s) \rangle \).

Note that in view of (b) above \( g_0^{-1}Eg_0 \cap \Gamma \) is Zariski dense in \( g_0^{-1}Eg_0 \). Therefore, \( F := g_0^{-1}Eg_0 \subset \mathcal{R}_{k_T/l_T}(G) \), \( l_T \) as in (a), and \( g_0 \) as in (b) satisfy the claims in the theorem. \( \square \)

Let \( k' \subset k_w \) be a closed subfield and assume fixed some \( \iota : SL_2 \times k' k_w \to G_w \). Put \( H = \iota(SL_2(k')) \).

**Corollary 5.10.** The conclusion of Theorem 5.9 holds for any probability measure \( \mu \) which is \( H \)-invariant and ergodic.
Proof. This is a direct corollary of Theorem 5.9, see the proof of Corollary 1.2.

We conclude the paper with the following lemma.

Lemma 5.11. Let the notation be as in Theorem 5.9 and its proof. Assume further that there is an \( l_T \)-representation \( (\rho, \Phi) \) and a vector \( q \in \Phi \) so that
\[
E = \{ g \in \mathcal{R}_{k_T \setminus l_T}(\mathbb{G}) : \rho(g)q = q \}.
\]
Then \( \mu \) is invariant under \( H^+ \).

Proof. Indeed we need to show that \( H^+ \subset \Sigma \). Note first that since \( SU \subset E \), we have that \( \rho(SU)q = q \). This together with Lemma 5.5(2) and the fact that \( U \subset U \) implies \( \rho(H^+)q = q \). In view of (5.18) we thus get that \( H^+ \subset E \), and hence
\[
W^{-}_H(s) \subset W^{-}_E(s) \subset E'.
\]
Therefore, \( H^+ = \langle W^{-}_H(s), W^{+}_H(s) \rangle \subset \Sigma \) as was claimed.

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