1) Find the vector projection of \( a = 2i - 3j + k \) onto \( b = i + 6j - 2k \).

   The vector projection of a vector \( a \) onto \( b \) is given by:
   
   \[
   \text{proj}_b a = \left( \frac{b \cdot a}{||b||^2} \right) b .
   \]

   In our case, \( a = (2, -3, 1) \) and \( b = (1, 6, -2) \).

   Then \( b \cdot a = 2 - 18 - 2 = -18 \)
   \[||b||^2 = 1^2 + 6^2 + (-2)^2 = 41\]

   Thus \( \text{proj}_b a = -\frac{18}{41}(1, 6, -2) = \left( -\frac{18}{41}, \frac{-18 \times 6}{41}, \frac{36}{41} \right) \)

2) A woman walks due west on the deck of a ship at 3 mph. The ship is moving north at a speed of 22 mph. Find the speed and direction of the woman relative to the water.

   \[
   \text{Speed} = \sqrt{22^2 + 3^2} = \sqrt{493}
   \]

   Direction is given by \( \langle -3, 22 \rangle \)

3) Find the angle between the vectors \( a = (\sqrt{3}, 1) \) and \( b = (0, 5) \).

   We are looking for \( \theta \) in the picture below.

   \[
   a \cdot b = ||a|| ||b|| \cos \theta \]

   Then \( a \cdot b = 0 \times \sqrt{3} + 5 \times 1 = 5 \)

   \[
   ||a|| = \sqrt{3 + 1^2} = \sqrt{4} = 2
   \]

   \[
   ||b|| = \sqrt{0^2 + 5^2} = \sqrt{25} = 5
   \]

   Thus \( \frac{s = 2 \times 5 \cos \theta}{\cos \theta = \frac{1}{2}} \).

   As \( 0 < \theta < \frac{\pi}{2} \) we conclude \( \theta = \frac{\pi}{3} \)
4) Find parametric equations for the line segment from (10,3,1) to (5,6,-3).

The general equation parametric of the line is given by
\[ r(t) = t \cdot \vec{d} + \vec{p} \] where \( \vec{d} \) is the direction and \( \vec{p} \) is a point in the line.

Let \( \vec{a} = (10,3,1) \) and \( \vec{b} = (5,6,-3) \)
I will pick \( \vec{a} \) to be my point on the line (one can also pick \( \vec{b} \) and the direction is \( \vec{ab} \) (one could also pick \( 5\vec{a} \))
Thus \( \vec{p} = \vec{a} = (10,3,1) \) and \( \vec{d} = \vec{ab} = \vec{b} - \vec{a} = (5,6,-3) - (10,3,1) \)
Thus one parametric representation of the line is given by
\[ r(t) = t (-5,3,-4) + (10,3,1) = (-5t+10, 3t+3, -4t+1) \] or
\[ \begin{cases} x(t) = -5t + 10 \\ y(t) = 3t + 3 \\ z(t) = -4t + 1 \end{cases} \]

5) Circle the correct answer.

|   | a) For any vectors \( \vec{a} \) and \( \vec{b} \), \( |\vec{a} \cdot \vec{b}| = (|\vec{a}|)(|\vec{b}|) \). | True | False |
|---|------------------------------------------------------------------|------|-------|
|   | b) For any vectors \( \vec{a} \) and \( \vec{b} \), \( \vec{a} \times \vec{b} = -\vec{b} \times \vec{a} \). | True | False |
|   | c) For any vectors \( \vec{a}, \vec{b}, \) and \( \vec{c} \), \( \vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \times \vec{c} \). | True | False |
|   | d) For any vectors \( \vec{a} \) and \( \vec{b} \), if \( |\vec{a} \cdot \vec{b}| = (|\vec{a}|)(|\vec{b}|) \) then \( \vec{a} \) is parallel to \( \vec{b} \). | True | False |
|   | e) The volume of the parallelepiped formed by the vectors \( \vec{a}, \vec{b}, \) and \( \vec{c} \) is the magnitude of their scalar triple product: \( V = |\vec{a} \cdot (\vec{b} \times \vec{c})| \). | True | False |

\( \Box \) We know that \( \vec{a} \cdot \vec{b} = ||\vec{a}|| ||\vec{b}|| \cos \theta \). Applying the absolute value on both sides, we find
\[ |\vec{a} \cdot \vec{b}| = ||\vec{a}|| ||\vec{b}|| \cos \theta | \]
\[ = ||\vec{a}|| ||\vec{b}|| |\cos \theta | \quad \text{(As \( ||\vec{a}||, ||\vec{b}|| \geq 0 \))} \]
\[ \leq ||\vec{a}|| ||\vec{b}|| \quad \text{(As \( 0 \leq |\cos \theta| \leq 1 \))} \]

\( \Box \) Can be checked using the definition of cross-product.

\( \Box \) Take \( \vec{a} = (-4,0,3) \), \( \vec{b} = (2,-1,0) \), \( \vec{c} = (0,2,5) \)
Then \( \vec{a} \times (\vec{b} \times \vec{c}) = (30,40) \neq (-38,-15,6) = (\vec{a} \times \vec{b}) \times \vec{c} \)

\( \Box \) From \( \Box \) \( |\vec{a} \cdot \vec{b}| = ||\vec{a}|| ||\vec{b}|| \) only in the case \( 1 \cos \theta | = 1 \)
or \( \theta = \pm \pi \), \( n = 0, 1, 2, \ldots \), which happens only if the vectors are parallel.

\( \Box \) Just the formula.
6) Find the equation of the plane through the points \((3, -1, 2), (8, 2, 4), \) and \((-1, -2, -3)\).

The general formula for the plane is given by:

\[ \mathbf{N} \cdot (\mathbf{x} - \mathbf{P}) = 0 \]

where \( \mathbf{N} \) is the normal vector.

Let \( \mathbf{P} = (3, -1, 2), \mathbf{Q} = (8, 2, 4), \mathbf{R} = (-1, -2, -3) \)

I will pick \( \mathbf{P} = (3, -1, 2) \) to be my point on the formula.

The normal vector is simultaneously perpendicular to \( \mathbf{PQ} \) and \( \mathbf{PR} \), thus is given by:

\[ \mathbf{N} = \mathbf{PQ} \times \mathbf{PR} = (5, 3, 2) \times (-4, -1, -5) \]

So \( (-13, 17, 7) \cdot (x-3, y+1, z-2) = 0 \)

\[ -13(x-3) + 17(y+1) + 7(z-2) = 0 \]

\[ -13x + 17y + 7z + 42 = 0 \]

7) Differentiate \( \mathbf{r}(t) = t \mathbf{a} + (b + tc) \).

Recall that for general vector \( \mathbf{d}, \mathbf{e}, \mathbf{f} \)

\[ \mathbf{d} \times (\mathbf{e} + \mathbf{f}) = \mathbf{d} \times \mathbf{e} + \mathbf{d} \times \mathbf{f} \]

Let \( \mathbf{d} = t \mathbf{a} \)

Then \( \mathbf{r}(t) = t \mathbf{a} + (b + tc) \)

\[ t(\mathbf{a} \times (b + tc)) \]

\[ \mathbf{r}'(t) = (a \times b) + 2t(a \times c) \]

8) Find the unit tangent vector for \( \mathbf{r}(t) = \cos t \mathbf{i} + 3t \mathbf{j} + 2 \sin 2t \mathbf{k} \) at the point where \( t = 0 \).

\[ \mathbf{r}(t) = \cos t \mathbf{i} + 3t \mathbf{j} + 2 \sin 2t \mathbf{k} \]

\[ = (\cos t, 3t, 2 \sin 2t) \]

So \( \mathbf{r}'(t) = (- \sin t, 3, 4 \cos 2t) \)

and \( \mathbf{r}'(0) = (0, 3, 4) \)

The unit vector is given by:

\[ \frac{\mathbf{r}'(0)}{||\mathbf{r}'(0)||} = \frac{1}{\sqrt{13^2 + 4^2}} (0, 3, 4) \]

\[ = \frac{1}{5} (0, 3, 4) = (0, \frac{3}{5}, \frac{4}{5}) \]
9) Evaluate \( \int (\cos \pi t + \sin \pi j + tk) dt \).

\[
\int (\cos \pi t \, i + \sin \pi t \, j + t \, k) \, dt = \int (\cos \pi t , \sin \pi t , t) \, dt
\]

\[
= \left( \int \cos \pi t \, dt , \int \sin \pi t \, dt , \int t \, dt \right)
\]

\[
= \left( \frac{\sin \pi t}{\pi} + k_1 , -\frac{\cos \pi t}{\pi} + k_2 , \frac{t^2}{2} + k_3 \right)
\]

where \( k_1, k_2, k_3 \) are constants of integration.

10) Circle the correct answer.

a) Find a parameterization of the circle having radius 4 and center \((-3, -2, 1)\) that lies in a plane parallel to the \(yz\)-plane.

(i) \( r(t) = (-3 + 4 \cos t, -2, 1 - 4 \sin t) \)

(ii) \( r(t) = (3 - 4 \sin t, 2 - 4 \cos t, -1) \)

(iii) \( r(t) = (3 - 4 \cos t, 2 - 1 + 4 \sin t) \)

(iv) \( r(t) = (3, 2 - 4 \cos t, -1 + 4 \sin t) \)

(v) \( r(t) = (-3 + 4 \sin t, -2 + 4 \cos t, 1) \)

(vi) \( r(t) = (-3, -2 + 4 \cos t, 1 - 4 \sin t) \)

b) Find an equation for the surface obtained by rotating the parabola \( z = y^2 \) about the \(z\)-axis.

(i) \( y^2 + x^2 = z \)

(ii) \( y^2 - x^2 = z \)

(iii) \( z^2 - x^2 = y \)

(iv) \( x^2 - y^2 = z \)

If the circle is parallel
to \(yz\)-plane, \(x(t)\) must be a constant. Thus the only options available are (i) and (iv). Now the equation of the circle is

\[
(\hat{y} \cdot \hat{z})^2 + (\hat{z} - 1)^2 = 16
\]

This must hold in the correct answer.

For (i) we have

\[
(y(t) + 2)^2 + (z(t) - 1)^2 = (4 \cdot 4 \cos t)^2 + (4 \cdot 4 \sin t)^2
\]

which is not constant. So we eliminate (i) and (iv). For (iv), \( (\hat{x} \cdot \hat{z})^2 + (\hat{z} - 1)^2 = 16 \), and the picture describes the object we are looking for. Note that for fixed \(z\), we need to have a circle, i.e.

\( x^2 + y^2 = z \) is the only option for which this is true.
1) Find the arc-length of the curve \( r(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + t \mathbf{k} \) from the point \((1, 0, 0)\) to \((1, 0, 2\pi)\).

The arc length of a curve \( r(t) \) is given by

\[
L = \int_{t_0}^{t_1} \| r'(t) \| \, dt.
\]

Now

\[
\| r'(t) \| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}
\]

At the point \((1, 0, 0)\), \(t = 0\), thus \(t_0 = 0\).

At the point \((1, 0, 2\pi)\), \(t = 2\pi\), thus \(t_1 = 2\pi\).

Hence

\[
L = \int_{0}^{2\pi} \sqrt{2} \, dt = \sqrt{2} (2\pi)
\]

2) The position vector of an object moving in a plane is given by \( r(t) = t^2 \mathbf{i} + t^3 \mathbf{j} \). Find its velocity, speed and acceleration when \( t = 1 \).

Velocity is given by \( r'(t) \), which in this case is

\[
r'(t) = 2t \mathbf{i} + 3t^2 \mathbf{j}.
\]

At \( t = 1 \), the velocity is

\[
r'(1) = 2 \mathbf{i} + 3 \mathbf{j}.
\]

Speed at \( t = 1 \) is given by \( \| r'(t) \| = \sqrt{2^2 + 3^2} = \sqrt{13} \).

Acceleration is given by \( r''(t) \). In our case

\[
r''(t) = 6 \mathbf{i} + 6t \mathbf{j}.
\]

and \( r''(1) = 6 \mathbf{i} + 6 \mathbf{j} \).

3) A moving particle starts at an initial position \( r(0) = (1, 0, 0) \) with initial velocity \( v(0) = \mathbf{i} - \mathbf{j} + \mathbf{k} \). Its acceleration is \( a(t) = 4t \mathbf{i} + 6t \mathbf{j} + \mathbf{k} \). Find its velocity and position at time \( t \).

To find \( v(t) \) we integrate \( a(t) \):

\[
v(t) = \int a(t) \, dt = \left( \int 4t \, dt, \int 6t \, dt, \int dt \right) + (c_1, c_2, c_3)
\]

so

\[
v(t) = \left( \frac{4t^2}{2} \mathbf{i} + \frac{6t^2}{2} \mathbf{j} + t \mathbf{k} + (c_1, c_2, c_3) \right)
\]

\[
= \left( \frac{4t^2 + c_1}{2} \mathbf{i} + \frac{6t^2 + c_2}{2} \mathbf{j} + (t + c_3) \mathbf{k} \right).
\]

To find \((c_1, c_2, c_3)\) we use that \( v(0) = \mathbf{i} - \mathbf{j} + \mathbf{k} \).

Then

\[
v(t) = \left( \frac{4t^2 + 1}{2}, \frac{6t^2 - 1}{2}, t + 1 \right).
\]

To find \( r(t) \), follow the same steps using

\[
r(t) = \int v(t) \, dt + (k_1, k_2, k_3)
\]

and to find \( k_1, k_2, k_3 \) use that \( r(0) = (1, 0, 0) \).