Test 3

Question 4: Show that

\[ \lim_{(x,y) \to (0,0)} \frac{x^2 - y^2}{x^2 + y^2} \]

does not exist.

A: Your job is to find two possible paths taking \((x, y)\) to \((0, 0)\) for which the function approaches two different values (i.e., two different paths for which the limit above disagree), yielding that the limit does not exist. Take \(x = y\) as \(x \to 0\). Then \((x, y) \to (0, 0)\), and in this path we have that \(\frac{x^2 - y^2}{x^2 + y^2} = \frac{x^2 - x^2}{x^2 + x^2} = \frac{0}{2x^2} = 0\) for any \((x, y)\) such that \(x = y, x \neq 0\). Thus \(\lim_{(x,x) \to (0,0)} \frac{x^2 - y^2}{x^2 + y^2} = 0\). On the other hand if we pick the \(y \to 0\) and \(x \to 0\), we have \(\frac{x^2 - y^2}{x^2 + y^2} = \frac{-y^2}{y^2} = -1\), for any \((x, y)\) such that \(x = 0\) and \(y \neq 0\). Therefore, \(\lim_{y \to 0}(0,y)\frac{x^2 - y^2}{x^2 + y^2} = -1\). (You could also have picked \(x \to 0\) and \(y = 0\). In this path, the limit of the function is 1 - check that as an exercise).
Question 5: True or False

a) The curvature is \( \kappa(t) = \frac{||r''(t)\times r'(t)||}{||r'(t)||^3} \). TRUE.

b) The unit tangent vector and the principle normal vector at a point on a vector curve are perpendicular. TRUE.

c) The gradient at a point on a level curve is perpendicular to the level curve. TRUE.

d) It is always true that \( f_{xy} = f_{yx} \). FALSE, a sufficient conditions is that the functions are continuous.

e) The normal line to a surface \( S \) at a point \( P \) is the line passing through \( P \) and perpendicular to the tangent plane at \( P \). TRUE.
Question 6: Find an equation of the plane tangent to the surface \( z = 4x^2 - y^2 + 2y \) at the point \((-1, 2, 4)\).

A: Consider an equation \( F(x, y, z) = 0 \). Then the plane to the surface defined by the equation at a point \((x_0, y_0, z_0)\) is given by

\[
F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0.
\]

In our case we were given \( z = 4x^2 - y^2 + 2y \), thus making a small adjustment our equation is

\[
z - 4x^2 - y^2 - 2y = 0.
\]

Hence \( F(x, y, z) = z - 4x^2 - y^2 - 2y \). We are asked the plane tangent to this surface at the point \((-1, 2, 4)\). Thus, we need to compute \( F_x(-1, 2, 4) \), \( F_y(-1, 2, 4) \) and \( F_z(-1, 2, 4) \). Note that

\[
F_x = -8x \\
F_y = -2y - 2 \\
F_z = 1
\]

Hence,

\[
F_x(-1, 2, 4) = 8 \\
F_y(-1, 2, 4) = -6 \\
F_z(-1, 2, 4) = 1
\]

and the equation of the plane is given by

\[
2(x + 1) - 6(y - 2) + 1(z - 4) = 0
\]

which is equivalent to

\[
2x - 6y + z + 10 = 0
\]
Question 7: Find the second order partial derivatives of $f(x, y) = x^3 + x^2y^3 - 2y^2$

A: The second-order derivatives are just $f_{xx}$, $f_{xy}$, $f_{yx}$ and $f_{yy}$. Since the function is continuous we have that $f_{xy} = f_{yx}$ (you can save time by just computing 3 partials instead of 4). In our case, $f_x = 3x^2 + 2xy^3$ and $f_y = 3x^2y^2 - 4y$. Therefore,

$$
\begin{align*}
    f_{xx} &= 6x + 2y^3 \\
    f_{yy} &= 6x^2y - 4 \\
    f_{xy} &= 6xy^2
\end{align*}
$$

(Check that, indeed, $f_{xy} = f_{yx}$.)
Question 8: If \( z = e^x \sin y \), where \( x = st^2 \) and \( y = s^2t \), find \( \frac{\partial z}{\partial s} \) and \( \frac{\partial z}{\partial t} \).

A: For a function \( z = f(x(s,t), y(s,t)) \) we have

\[
\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} \tag{1}
\]
\[
\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t} \tag{2}
\]

In our case, \( f(x(s,t), y(s,t)) = e^x \sin y \), \( x(t,s) = st^2 \) and \( y(t,s) = s^2t \) so

\[
\frac{\partial z}{\partial x} = e^x \sin y \tag{3}
\]
\[
\frac{\partial z}{\partial y} = e^x \cos y \tag{4}
\]
\[
\frac{\partial x}{\partial s} = t^2 \tag{5}
\]
\[
\frac{\partial y}{\partial s} = 2st \tag{6}
\]
\[
\frac{\partial x}{\partial t} = 2st \tag{7}
\]
\[
\frac{\partial y}{\partial t} = s^2 \tag{8}
\]

Plugging (3) – (8) into (1) and (2) we find:

\[
\frac{\partial z}{\partial s} = e^x \sin(y)t^2 + 2e^x \cos(y)st
\]
\[
\frac{\partial z}{\partial t} = 2e^x \sin(y)st + e^x \cos(y)s^2
\]
Question 9: Find the local maximum and minimum values and saddle points of \( f(x, y) = x^4 + y^4 - 4xy + 1 \).

A: The question asks to find the critical points and say what they are. Setting up the first derivatives to zero gives us the critical point. The second derivative test tells us what they are.

The derivatives are

\[
\begin{align*}
  f_x(x, y) &= 4x^3 - 4y \\
  f_y(x, y) &= 4y^3 - 4x
\end{align*}
\]

\( f_x(x, y) = 0 \) and \( f_y(x, y) = 0 \) simultaneously when \( x^3 = y \) (1) and \( y^3 = x \) (2). If we plug in (2) into (1) we conclude that \( y^9 = y \), which is only true if \( y = 1 \), \( y = 0 \) or \( y = -1 \). Thus the critical points are \((0, 0)\), \((1, 1)\) and \((-1, -1)\).

Now let’s perform the second derivative test. At a critical point \((a, b)\) compute \( D \) as follows

\[
D(a, b) = f_{xx}(a, b) - f_{yy}(a, b) - (f_{xy}(a, b))^2.
\]

If \( D(a, b) > 0 \) and \( f_{xx}(a, b) > 0 \), then \( f(a, b) \) is a local minimum. If \( D(a, b) > 0 \) and \( f_{xx}(a, b) < 0 \) then \( f(a, b) \) is a local maximum. If \( D(a, b) < 0 \), then \( f(a, b) \) is a saddle point.

Remark: If \( D(a, b) = 0 \) the test gives no information.

Now if we compute the second-order partial derivatives we find \( f_{xx}(x, y) = 12x^2 \), \( f_{yy}(x, y) = 12y^2 \) and \( f_{xy}(x, y) = -4 \). Then

\[
\begin{align*}
D(0, 0) &= f_{xx}(0, 0)f_{yy}(0, 0) - f_{xy}(0, 0)^2 \\
&= -16 \\
D(1, 1) &= (12) \times (12) - 16 \\
&= 128 \\
D(-1, -1) &= (12) \times (12) - 16 \\
&= 128
\end{align*}
\]

Hence, \((0, 0)\) is a saddle point, \((1, 1)\) is a local minimum, and \((-1, -1)\) is another point of local minimum.
Question 10: a) Find the maximum slope on the graph of \( f(x, y) = 2 \sin(xy) \) at the point \( P(0, 4) \). b) Find the directional derivative of \( f(x, y) = \sqrt{6x + 3y} \) at the point \( (2, 2) \) in the direction \( v = \mathbf{i} + \mathbf{j} \).

a) A: Finding the maximum slope of the graph corresponds to maximize the directional derivative. By theorem 15 (section 15.6) it takes place when the direction is the same as the gradient and is maximum is \( ||\nabla f|| \). Computing the gradient we find \( \nabla f(x, y) = \langle 2y \cos(xy), 2x \cos(x, y) \rangle \), so \( \nabla f(0, 4) = \langle 8, 0 \rangle \), and \( ||\nabla f(0, 4)|| = \sqrt{8^2} = 8 \). (alternative (ii)).

b) A: The directional derivative is given by

\[
\nabla f \cdot \frac{v}{||v||}
\]

In our case, \( \nabla f(x, y) = \langle \frac{6}{2\sqrt{6x+3y}}, \frac{3}{2\sqrt{6x+3y}} \rangle \) for all \( x, y \), in particular, \( \nabla f(2, 2) = \langle \frac{2}{2\sqrt{2}}, \frac{1}{2\sqrt{2}} \rangle \).

\( v = \mathbf{i} + \mathbf{j} \) or \( v = \langle 1, 1 \rangle \) and so \( ||v|| = \sqrt{1^2 + 1^2} = \sqrt{2} \). Therefore, \( \frac{v}{||v||} = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle \), and so

\[
\nabla f \cdot \frac{v}{||v||} = \langle \frac{2}{2\sqrt{2}}, \frac{1}{2\sqrt{2}} \rangle \cdot \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle = \frac{3}{4} \) (alternative (iii)).
**Test 4A**

**Question 16:** Determine the minimum value of \( f(x, y) = 3x - 4y + 3 \) subject to the constraint \( g(x, y) = x^2 + y^2 - 1 = 0 \).

A: Following the method of Lagrange multiplier we need to find the values of \((x, y)\) and \(\lambda\) such that

\[
\nabla f(x, y) = \lambda \nabla g(x, y)
\]

with

\[
g(x, y) = 0.
\]

In our case, \( \nabla f(x, y) = \langle 3, -4 \rangle \) and \( \nabla g(x, y) = \langle 2x, 2y \rangle \). Then we need to find \((x, y)\) and \(\lambda\) such that \(3\lambda = 2x \) (1), \(-4\lambda = 2y \) (2) and \(x^2 + y^2 = 1 \) (3). Let’s solve for \(\lambda\): From (1), \(x = \frac{3\lambda}{2}\) and from (2), \(y = -2\lambda\). Now using (3), \(\frac{9}{4}\lambda^2 + 4\lambda^2 = 1\), so \(\lambda^2 = \frac{4}{25}\) or \(\lambda = +/- \frac{2}{5}\). It follows that \(x = +/- \frac{3}{5}\) and \(y = +/- \frac{4}{5}\).

Thus we need to evaluate the function at \((\frac{3}{5}, -\frac{4}{5})\) and \((-\frac{3}{5}, \frac{4}{5})\), so \(f\left(\frac{3}{5}, -\frac{4}{5}\right) = \frac{9+16+3}{5} = \frac{28}{5}\) and \(f\left(-\frac{3}{5}, \frac{4}{5}\right) = \frac{-9-16+3}{5} = -\frac{22}{5}\). Thus the miminum is \(-\frac{22}{5}\) and it’s achieved when \((x, y) = (-\frac{3}{5}, \frac{4}{5})\).
Question 17: Evaluate the integral \( I = \int_1^3 \int_1^3 \frac{x}{y} + \frac{y}{x} \, dy \, dx \).

A: We first compute \( I_1 = \int_1^3 \frac{x}{y} + \frac{y}{x} \, dy \) treating \( x \) as a constant. We find

\[
I_1 = \int_1^3 \frac{x}{y} + \frac{y}{x} \, dy
= -x \ln y \bigg|_1^3 + \frac{y^2}{2x} \bigg|_1^3
= x [\ln 3 - \ln 1] + \frac{1}{x} \left[ \frac{9}{2} - \frac{1}{2} \right]
= x \ln 3 + \frac{4}{x}.
\]

Next we compute \( I = \int_1^3 I_1 \, dx \),

\[
I = \int_1^3 x \ln 3 + \frac{4}{x} \, dx
= \ln 3 \frac{x^2}{2} \bigg|_1^3 + 4 \ln x \bigg|_1^3
= 4 \ln 3 + 4 \ln 3
= 8 \ln 3.
\]
Question 18: Evaluate $I = \int_0^1 \int_{y^2}^1 2y \cos(x^2) \, dx \, dy$ by changing the order of integration.

A: The region one wants to integrate is given by $D = \{(x, y) : 0 \leq y \leq 1; y^2 \leq x \leq 1\}$ which is equivalent to $D = \{(x, y) : 0 \leq x \leq 1; 0 \leq y \leq \sqrt{x}\}$.

Thus

$$I = \int_0^1 \int_{y^2}^1 2y \cos(x^2) \, dx \, dy = \int \int_D 2y \cos(x^2) \, dx \, dy$$
$$= \int_0^1 \int_0^{\sqrt{x}} 2y \cos(x^2) \, dy \, dx$$
$$= \int_0^1 y^2 \cos(x^2) \bigg|_0^{\sqrt{x}} \, dx$$
$$= \int_0^1 x \cos(x^2) \, dx$$
$$= \int_0^1 \frac{1}{2} \cos(u) \, du$$
$$= \frac{1}{2} \sin(u) \bigg|_0^1$$
$$= \frac{1}{2} \sin(1)$$

(By making $u = x^2$)