1. Solutions

(3.3, 5) Let $A$ and $B$ be $n \times n$ matrices.
(a) Show that $A$ is nonsingular if and only if $A^T$ is nonsingular.
(b) Show that $\det(AB) = \det(BA)$.

Proof. For the first one, since $A$ is nonsingular, then $\det(A) \neq 0$. But since $\det(A^T) = \det(A)$, this implies that $\det(A^T) \neq 0$ and so $A^T$ is nonsingular. For the second,
$$
\det(AB) = \det(A) \det(B) = \det(B) \det(A) = \det(BA).
$$

□

(3.3, 7) Let $A$ and $B$ be $n \times n$ matrices.
(a) Show that $\det(AA^T) \geq 0$.
(b) Show that $\det(AB^T) = \det(A^T) \det(B)$.

Proof. For the first one,
$$
\det(AA^T) = \det(A) \det(A^T) = \det(A) \det(A) = (\det(A))^2.
$$
The result follows from the fact that the square of any real number is nonnegative. For the second one,
$$
\det(AB^T) = \det(A) \det(B^T) = \det(A^T) \det(B^T) = \det(A^T) \det(B).
$$

□

(3.3, 12) Suppose that $\det(A)$ is an integer.
(a) Prove that $\det(A^n)$ is not prime, for $n \geq 2$.
(b) Prove that if $A^n = I$ for some $n \geq 1$, $n$ odd, then $\det(A) = 1$.

Proof. Since $\det(A^n) = \det(A)^n$, we know that $\det(A)$ divides $\det(A^n)$. This proves the first statement. For the second, $A^n = I$ implies that $\det(A^n) = \det(A)^n = 1$. Therefore, $\det(A) = \pm 1$, and when $n$ is odd, $\det(A)$ must be 1.

□

(3.3, 15) If all entries of a square matrix $A$ are integers and $\det(A) = \pm 1$, show that all entries of $A^{-1}$ are integers.

Proof. The fact that $\det(A) = \pm 1$ implies that when we perform Gaussian elimination on $A$, we never have to multiply rows by scalars. This means that for each column, the pivot entry is created by the previous column’s row operations and can be brought into place by swapping rows. (And
the first column must already contain a 1.) Therefore, we never need to multiply by a non-integral value to perform Gaussian elimination.

\[ \Box \]

(3.4, 7) Let \( A \) be a diagonalizable \( n \times n \) matrix.

(a) Show that \( A \) has a cube root — that is, that there is a matrix \( B \) such that \( B^3 = A \).

(b) Give a sufficient condition for \( A \) to have a square root.

Proof. Since \( A \) is diagonalizable, we have the equation \( A = PDP^{-1} \) for a diagonal matrix \( D \). In order to find fractional powers of \( A \), it suffices to find them for \( D \), which can be done elementwise. Thus, \( A \) always has a cube root and has a square root when all the entries of \( D \) are positive. \[ \Box \]

(3.4, 12) Let \( A \) be an upper triangular \( n \times n \) matrix.

(a) Prove that \( \lambda \) is an eigenvalue for \( A \) if and only if \( \lambda \) appears on the main diagonal of \( A \).

(b) Show that the algebraic multiplicity of an eigenvalue \( \lambda \) of \( A \) equals the number of times \( \lambda \) appears on the main diagonal.

Proof. Both parts follow from the fact that the determinant of an upper triangular matrix is the product of the terms on the diagonal; if \( A \) is upper triangular, so is \( \lambda I - A \), with diagonal terms \( (\lambda - a_{ii}) \).

\[ \Box \]

(3.4, 13) Let \( A \) be an \( n \times n \) matrix. Prove that \( A \) and \( A^T \) have the same characteristic polynomial.

Proof. This follows immediately from the observation that \( (\lambda I - A)^T = (\lambda I - A^T) \). (Note that this is not a general fact about sums, of course, but follows by inspection here.) \[ \Box \]

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