Instructions: No calculators or notes allowed. Show all work. If extra space is required, please use the back of the page.

1. (20 points) Find the distance from the point \( P = (2, -2, 1) \) to the plane \( 3x - 2y + 8z = 1 \).

The point \((x_0, y_0, z_0) = (1, 5, 1)\) is on the plane. \( P = (x_1, y_1, z_1) = (2, -2, 1) \)

The distance from a point to a plane \( Ax + By + Cz = D \) is given by

\[
D = \frac{|A(x_1-x_0) + B(y_1-y_0) + C(z_1-z_0)|}{\sqrt{A^2+B^2+C^2}} = \frac{|3(2-1) - 2(-2-5) + 8(1-1)|}{\sqrt{3^2+2^2+8^2}}
\]

\[
= \frac{|13 + 14|}{\sqrt{77}} = \frac{27}{\sqrt{77}}.
\]

2. (20 points) Determine a unit normal vector to the hyperboloid \( x^2 + y^2 - z^2 = 19 \) at \((2, 4, -1)\).

The gradient of a function is normal to its level surfaces.

\[ f(x,y,z) = x^2 + y^2 - z^2 \]

\[ \nabla f(x,y,z) = (2x, 2y, -2z) \]

\[ \nabla f(2,4,-1) = (4, 8, 2) \text{ is normal to the surface } x^2 + y^2 - z^2 = 19. \]

\[ \vec{n} = \frac{(4, 8, 2)}{\sqrt{4^2+8^2+2^2}} = \frac{1}{\sqrt{84}} (4, 8, 2) = \frac{1}{12} (2, 4, 1) \]
3. (20 points) Maximize the function \( f(x, y, z) = x + z \) subject to the constraint \( x^2 + y^2 + z^2 = 1 \). That is, the solution satisfies the system of equations: \( \nabla f = \lambda \nabla g \) and \( g = 1 \). Since \( \nabla g = (2x, 2y, 2z) \), the system of equations is: \( 2\lambda x = 1 \), \( 2\lambda y = 0 \), \( 2\lambda z = 1 \), \( x^2 + y^2 + z^2 = 1 \). Since \( \nabla f = (1, 0, 1) \). Inserting the first three equations into the last gives \( (\frac{1}{2\lambda})^2 + 0^2 + (\frac{1}{2\lambda})^2 = 1 \) \( \Rightarrow \lambda = \pm \frac{1}{2\lambda} \), \( x = \frac{1}{2\lambda} \), \( y = 0 \), \( z = \frac{1}{2\lambda} \) gives two constrained extrema \( \left( \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right) \) and \( \left( -\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}} \right) \) with values:

\[
\begin{align*}
&f\left( \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right) = \frac{2}{\sqrt{2}} = \sqrt{2}, \text{ maximum value} \\
&f\left( -\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}} \right) = -\frac{2}{\sqrt{2}} = -\sqrt{2}, \text{ minimum value}
\end{align*}
\]

4. (20 points) Let \( \mathbf{G}(x, y) = (xe^{x^2+y^2} + 2xy)i + (ye^{x^2+y^2} + x^2)j \). Determine if \( \mathbf{G} \) is a gradient vector field. If it is, find a scalar function \( f \) with \( \nabla f = \mathbf{G} \).

If \( \mathbf{G} = (P, Q) = \nabla f = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) \) then \( \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \Rightarrow \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \).

\[
\frac{\partial Q}{\partial x} = y(2x)e^{x^2+y^2} + 2x \quad \frac{\partial P}{\partial y} = x(2y)e^{x^2+y^2} + 2x.
\]

Since \( \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \), \( \mathbf{G} \) is a gradient vector field. To find \( f \), we integrate

\[
\begin{align*}
f &= \int P \, dx = \int (xe^{x^2+y^2} + 2xy) \, dx = \frac{1}{2}e^{x^2+y^2} + xy + h_1(y) \\
f &= \int Q \, dy = \int (ye^{x^2+y^2} + x^2) \, dy = \frac{1}{2}e^{x^2+y^2} + xy + h_2(x)
\end{align*}
\]

which are compatible with \( f(x, y) = \frac{1}{2}e^{x^2+y^2} + xy \).
5. (20 points) Let \( \mathbf{H} = \mathbf{G} + (z - y)i + yk \), with \( \mathbf{G} \) defined as in the previous problem.
Evaluate the surface integral
\[
I = \int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{S},
\]
where \( S \) is the portion of the unit sphere \( x^2 + y^2 + z^2 = 1 \) with \( z \geq 1/2 \).

Apply Stokes Theorem: \( \int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{S} = \int_C \mathbf{H} \cdot d\mathbf{r} \) when \( C \) is the unit circle on the unit sphere with \( x = \frac{1}{2} \); namely \( y^2 + z^2 = \frac{3}{4} \). Since \( \nabla \times \mathbf{H} = \mathbf{G} \),
\[
I = \int_C (\mathbf{H} - \mathbf{G}) \cdot d\mathbf{r} = \int_C ((z-y)i + yk) \cdot d\mathbf{r}.
\]
Parameterize \( C \) with
\[
c(\theta) = \left( \frac{1}{2}, \frac{\sqrt{3}}{2} \cos \theta, \frac{\sqrt{3}}{2} \sin \theta \right), \quad 0 \leq \theta \leq 2\pi.
\]
\[
c'(\theta) = \left( 0, -\frac{\sqrt{3}}{2} \sin \theta, \frac{\sqrt{3}}{2} \cos \theta \right)
\]
\[
I = \int_0^{2\pi} \left( \frac{\sqrt{3}}{2} \sin \theta \cos \theta \right) \frac{\sqrt{3}}{2} \cos \theta \, d\theta = \int_0^{2\pi} \frac{3}{4} \sin \theta \, d\theta = \frac{3}{8} \left[ \cos \theta \right]_0^{2\pi} = \frac{3\pi}{4}
\]

since the integral of sine or cosine over a full period is zero.

6. (20 points) Let \( W \) be the three dimensional region under the graph of \( f(x, y) = \exp(x^2 + y^2) \) and over the region in the plane defined by \( 1 \leq x^2 + y^2 \leq 2 \). Find the volume of \( W \).

Use cylindrical coordinates:
\[
V = \text{Volume} = \int_{r=1}^{\sqrt{2}} \int_{\theta=0}^{2\pi} \int_{z=0}^{r \sqrt{2}} r \, dz \, d\theta \, dr = 2\pi \int_{r=1}^{\sqrt{2}} r \, e^{r^2} \, dr
\]

Change variables \( u = r^2 \) \( du = 2r \, dr \) \( \frac{1}{2} du = rdr \),
\[
V = 2\pi \int_{u=1}^{2} \frac{1}{2} e^u \, du = \pi \left( e^u \right)_{u=1}^{2} = \pi (e^2 - e)
\]
7. (20 points) Evaluate

\[ \int_0^1 \int_{\sqrt{x}}^{\infty} e^{x/y} \, dy \, dx. \]

Change order of integration.

\[ 0 \leq y \leq 1, \quad y^2 \leq x \leq y. \]

\[ u = \frac{x}{y}, \quad du = \frac{1}{y} \, dx, \quad dx = y \, du. \]

\[ I = \int_0^1 \int_{y^2}^{y} e^{x/y} \, dx \, dy = \int_0^1 \int_{y^2}^{y} e^{u} \cdot \frac{1}{y} \, du \, dy = \int_0^1 y \left( e^u \right)_{u=y^2}^{u=y} \, dy = \int_0^1 y \left( e^y - e^{y^2} \right) \, dy \]

\[ = e \int_0^1 y \, dy - \int_0^1 ye^y \, dy \]

Integrate by parts.

\[ U = y, \quad dV = e^y \, dy, \quad du = dy, \quad V = e^y \]

\[ = e \left[ \frac{1}{2} y^2 \right]_0^1 - \left[ ye^y \right]_0^1 + \int_0^1 e^y \, dy \]

\[ = \frac{1}{2} e - \left[ e - e \right] = \frac{1}{2} e - 1. \]

8. (20 points) Find the flux of the vector field \( \mathbf{F} = (2x - xy)i - yj + yz\mathbf{k} \) out of the region \( W \) in question 6.

Apply divergence theorem: \( \text{div} ( \mathbf{F} ) = 2 - y - 1 + y = 1 \). Use result of \#6.

\[ \iint_W \mathbf{F} \cdot d\mathbf{S} = \iiint_W \text{div} ( \mathbf{F} ) \, dV = \iiint_W 1 \, dV = \text{vol}(W) = \pi (e^2 - 1) \]
9. (20 points) Compute the mass of the wire described by the curve \( c(t) = (\sin t, \cos t, t) \), where \( 0 \leq t \leq \pi \), if the mass density of the wire is \( f(x, y, z) = x^2 + y^2 + z^2 \) grams per centimeter and lengths are measured in centimeters.

Use a path integral to calculate the mass. \( \mathbf{c}'(t) = (\cos t, -\sin t, 1) \)

\[
|| \mathbf{c}'(t) || = \sqrt{2}
\]

\[
\text{Mass} = \int_{S_c} f \, ds = \int_{t=0}^{\pi} \left( \sin^2 t + \cos^2 t + t^2 \right) || \mathbf{c}'(t) || \, dt
\]

\[
= \sqrt{2} \int_{t=0}^{\pi} \left( 1 + \frac{1}{3} t^3 \right) \bigg|_{t=0}^{t=\pi} = \sqrt{2} \left( \pi + \frac{1}{3} \pi^3 \right)
\]

10. (20 points) Let \( V \) be the region in the positive octant bounded by the sphere \( x^2 + y^2 + z^2 = 1 \). Calculate the surface integral

\[
\int \int_S \mathbf{F} \cdot d\mathbf{S}
\]

where

\[
\mathbf{F}(x, y, z) = (3x - z^4)i - (x^2 - y)j + (xy^2)k
\]

and \( S \) is the boundary of the set \( V \).

Apply the Divergence Theorem:

\[
\mathbf{I} = \iiint_V \mathbf{F} \cdot dV = \iiint_V \text{div}(\mathbf{F}) \, dV
\]

\[
\text{div}(\mathbf{F}) = 3 + 1 = 4
\]

\[
\mathbf{I} = 4 \cdot \text{vol}(V) = 4 \cdot \frac{1}{8} \left( \frac{1}{3} \pi (1)^3 \right) = \frac{2}{3} \pi \quad \text{using fact that vol}(V) \text{ is } \frac{1}{8} \text{ the volume of a sphere.}
11. (20 points) Let \( C \) be the boundary of the rectangle \([1,3] \times [2,3]\). Evaluate
\[
\int_C \left( \frac{2y + \sin x}{1 + x^2} \right) \, dx + \left( \frac{x + e^y}{1 + y^2} \right) \, dy.
\]
Apply Green's Theorem: \( \int_C P \, dx + Q \, dy = \int_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA \).
\[
\frac{\partial Q}{\partial x} = \frac{1}{1 + y^2}, \quad \frac{\partial P}{\partial y} = \frac{2}{1 + x^2}.
\]
\[
\int_R \left( \frac{1}{1 + y^2} - \frac{2}{1 + x^2} \right) \, dA = \int_{x=1}^{3} \int_{y=2}^{3} \left( \frac{1}{1 + y^2} - \frac{2}{1 + x^2} \right) \, dy \, dx = 2 \int_{x=1}^{3} \frac{1}{1 + y^2} \, dy - 2 \int_{x=1}^{3} \frac{1}{1 + x^2} \, dx
\]
\[
= 2 \left( \arctan(3) - \arctan(2) \right) - 2 \left( \arctan(3) - \arctan(1) \right)
\]
\[
= 2 \left( \arctan(1) - \arctan(2) \right)
\]

12. (20 points) Find the work done by the force, \( \mathbf{F} = -10r/r^6 \), moving from a distance \( r = 10 \) to a distance \( r = 9 \), along the path \( c(t) = 10t \mathbf{i} + 9(1 - t) \mathbf{k} \).

Note that \( \mathbf{F} \) is a gradient vector field. \( \mathbf{F}(x,y,z) = -10 \left( \frac{x,y,z}{\sqrt{x^2+y^2+z^2}} \right) \).

\[
\int \frac{x}{(x^2+y^2+z^2)^{3/2}} \, dx = \frac{u}{2} \left( \frac{y^2+z^2}{z^2} \right) = -\frac{1}{2} \left( \frac{x^2+y^2+z^2}{z^2} \right)^2
\]

\( \mathbf{F} = \nabla f \) where \( f = \frac{5}{2}r^4 \). Therefore, by the fundamental theorem of calculus for gradient vector fields,

\[
\text{Work} = f(9) - f(10) = \frac{5}{2} \left( \frac{1}{9^4} \right) - \frac{5}{2} \left( \frac{1}{10^4} \right).
\]