Instructions: No calculators or notes allowed. If extra space is required, please use the back of the page.

1. (20 points) Find the volume of the parallelepiped, with one corner at the origin, spanned by the vectors (1, 0, 1), (1, 1, 1), and (3, 2, 0).

\[
\text{Volume of a parallelepiped } = |\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})| \\
\mathbf{B} \times \mathbf{C} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
1 & 1 & 1 \\
3 & 2 & 0
\end{vmatrix} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
2 & 1 & 1 \\
3 & 0 & 1
\end{vmatrix}
\]

\[= -2\hat{i} + 3\hat{j} - \hat{k}\]

\[
\text{Volume } = \left| (1, 0, 1) \cdot (-2, 3, -1) \right| = |-2 - 1| = 3.
\]

2. (20 points) Compute the limit

\[
\lim_{(x,y) \to (0,0)} \frac{(x - y)^2}{x^2 + y^2}
\]

if it exists, or explain why the limit does not exist.

The limit does not exist. First take the limit along the line \( y = x \):

\[
\lim_{x \to 0} \frac{(x - x)^2}{x^2 + x^2} = \lim_{x \to 0} \frac{0}{2x^2} = 0.
\]

Next take the limit along the line \( y = -x \):

\[
\lim_{x \to 0} \frac{(x - (-x))^2}{x^2 + (-x)^2} = \lim_{x \to 0} \frac{4x^2}{2x^2} = \lim_{x \to 0} 2 = 2.
\]

Since the limits along the two contours are not equal, the limit \( \lim_{(x,y) \to (0,0)} \frac{(x - y)^2}{x^2 + y^2} \) does not exist.
3. (20 points) Find the tangent plane to the surface \( x^2 + y^2 - z^2 = 18 \) at the point \((3, 5, -4)\).

The surface is the level surface of the function \( f(x, y, z) = x^2 + y^2 - z^2 = 18 \). The tangent plane is \( \nabla f(3, 5, -4) \cdot (x-3, y-5, z+4) = 0 \).

\[ \nabla f = (2x, 2y, -2z) \quad \nabla f(3, 5, -4) = (6, 10, 8) . \] So the equation of the tangent plane is \((6, 10, 8) \cdot (x-3, y-5, z+4) = 0 \) or \( 6(x-3) + 10(y-5) + 8(z+4) = 0 \).

4. (20 points) Find and classify the critical points for \( H(x, y) = 2y^4 + 3x^2 - 8xy \).

Use the first derivative test to find the critical points

\[ \nabla H = (6x-8y, 8y^3-8x) = 0 . \] One solution is \( x=0, y=0 \).

Note if \( x=0 \) then \( y=0 \), and if \( y=0 \), then \( x=0 \) from the first equation.

\[ 6x - 8y = 0 \quad x = \frac{4}{3}y \]
\[ 8y^3 - 8x = 0 \quad x = y^3 \]

So \( y^3 = \frac{4}{3}y \) since \( y \neq 0 \), we can divide by \( y \)

\[ y^2 = \frac{4}{3} \quad \text{or} \quad y = \pm \frac{2}{\sqrt{3}} \]

which gives two critical points \((\frac{8}{3\sqrt{3}}, \frac{2}{\sqrt{3}})\)

and \((-\frac{8}{3\sqrt{3}}, -\frac{2}{\sqrt{3}})\). To classify the critical points we use the second derivative test.

\[ \frac{\partial^2 H}{\partial x^2} = 6, \quad \frac{\partial^2 H}{\partial x \partial y} = -8, \quad \frac{\partial^2 H}{\partial y^2} = 24y^2 . \]

\((0, 0)\) \( D = (6)(10) - (-8)^2 = -64 < 0 \) saddle point.

\((\frac{8}{3\sqrt{3}}, \frac{2}{\sqrt{3}})\) \( D = (6)(32) - (-8)^2 = 128 > 0 \) local min.

\((-\frac{8}{3\sqrt{3}}, -\frac{2}{\sqrt{3}})\) \( D = (6)(32) - (-8)^2 = 128 > 0 \) local min.
5. (20 points) The quantity of widgets that a company can produce is given by $Q(x, y) = xy$. The cost of production is $C(x, y) = 2x + 3y$. If the company can spend $C(x, y) = 10$, what is the maximum quantity of widgets that can be produced?

Use Lagrange multipliers. Need to solve $\nabla Q = \lambda \nabla C$ and $C = 10$.

$\nabla Q = (y, x)$  \hspace{0.5cm}  $\nabla C = (2, 3)$

$y = 2\lambda$
$x = 3\lambda$

$2x + 3y = 10$  \hspace{0.5cm}  so  \hspace{0.5cm}  $2(3\lambda) + 3(2\lambda) = 12\lambda = 10$  \hspace{0.5cm}  or  $\lambda = \frac{5}{6}$

Thus, $x = 3 \left( \frac{5}{6} \right) = \frac{5}{2}$  \hspace{0.5cm}  and  \hspace{0.5cm}  $y = 2 \left( \frac{5}{6} \right) = \frac{5}{3}$.

The maximum quantity of widgets that can be produced is $Q \left( \frac{5}{2}, \frac{5}{3} \right) = \left( \frac{5}{2} \right) \left( \frac{5}{3} \right) = \frac{25}{6}$.