Problem Set # 1
Multivariable Analysis (M375T)
Due: January 17

There are weekly homework assignments due each Thursday at the beginning of class. (This week you only have a few days, so the problem set is short.) Please work the problems neatly and staple your pages together. Please number the problems. Include your name at the top of your homework! Do not show scratch work. I encourage you to discuss the problems with classmates, friends, parents, etc. However, I expect you to write up your own solutions to the problems. Please come and discuss the problems (and the class generally) with me during office hours.

The problems are often open-ended to encourage exploration. Please be sure you can work the easier problems before tackling the hard ones. I will give some problems which are not directly related to the lectures.

1. Write careful proofs of the following theorems beginning from the definition of a vector space and a field, as given in lecture and recorded in the handout “Vector spaces” on the website. In each $V$ is a a vector space over a field $F$.
   (a) If $\xi \in V$, then $0 \cdot \xi = 0$. (The first ‘0’ is the additive identity in the field $F$; the second ‘0’ is the zero vector in the vector space $V$.)
   (b) The inverse $\eta$ of a vector $\xi$ is unique and equal to $(−1) \cdot \xi$, where $−1 \in F$ is the additive inverse of the multiplicative identity element $1 \in F$.

2. What is the dimension of the span of the vectors
   $\xi_1 = (1,1,1)$
   $\xi_1 = (0,1,−1)$
   $\xi_1 = (1,0,2)$
   in $\mathbb{R}^3$? Justify your answer.

3. Fix $a, b \in \mathbb{R}$ not both zero. Let $V$ be the real vector space of continuous functions $\mathbb{R} \to \mathbb{R}$, and $W$ the subspace spanned by the functions
   $e_1 = e^{ax} \cos(bx)$
   $e_2 = e^{ax} \sin(bx)$

   where $x$ is the coordinate on the domain $\mathbb{R}$. Write the matrix for the linear operator $d/dx$ on $W$ in the basis $e_1, e_2$. Compute the inverse of this matrix. What calculus formulas can you deduce from this linear algebra computation?
4. Continuing with the notation of the previous problem, prove that the three functions
\[ \sin x, \cos x, e^x \]
are linearly independent in \( V \).

5. This problem gives practice with index notation and the summation convention, which states: an index which is repeated, once as a subscript and once as a superscript, is summed over. Note carefully the placement (superscript vs. subscript) of the indices in what follows. The actual name of the index (\( i \) or \( j \) or \( \mu \)) is arbitrary, though as always a judicious choice of notation helps you and your readers.

Let \( V \) be an \( n \)-dimensional vector space. Suppose \( \{ e_j \} \) and \( \{ f_i \} \) are two bases for \( V \) which are related by the equation
\[ e_j = P^i_j f_i, \tag{*} \]
where \( P \) is an invertible matrix. My convention is that \( i \) is the row index and \( j \) the column index when we view \( P^i_j \) as the entry in a matrix.

(a) Suppose \( \xi \in V \) is a vector. Then we can find real numbers \( \xi^j \) and \( \tilde{\xi}^i \) such that \( \xi = \xi^j e_j = \tilde{\xi}^i f_i \).

Express \( \tilde{\xi}^i \) in terms of the \( \xi^j \) by substituting (\( * \)) and using the uniqueness of the expansion of a vector in terms of a basis.

(b) Suppose \( T: V \to V \) is a linear transformation. Relative to the basis \( \{ e_j \} \) it is expressed as the matrix \( A \) defined by \( T e_j = A^i_j e_i \), and relative to the basis \( \{ f_i \} \) it is expressed as the matrix \( B \) defined by \( T f_i = B^j_i f_j \). What is the relationship between \( A \) and \( B \)?

(c) The dual space \( V^* \) is the vector space of all linear functionals \( V \to \mathbb{R} \); it is also \( n \) dimensional. Every basis of \( V \) gives rise to a dual basis of \( V^* \). For the basis \( \{ e_j \} \) of \( V \) the dual basis \( \{ e^i \} \) of \( V^* \) is defined by the equation
\[ e^i(e_j) = \delta^i_j = \begin{cases} 1, & i = j; \\ 0, & i \neq j. \end{cases} \]

(This equation defines the symbol \( \delta^i_j \).) The dual basis \( \{ f^j \} \) is defined similarly. Express \( f^j \) in terms of the \( e^i \).

(d) Suppose \( \omega \in V^* \). Then we define its components relative to the basis \( \{ e^i \} \) by the equation \( \omega = \omega^i e^i \) and its components relative to the basis \( \{ f^j \} \) by the equation \( \omega = \tilde{\omega}^j f^j \). Express the \( \omega_i \) in terms of the \( \tilde{\omega}_j \).

(e) Compute the evaluation \( \omega(\xi) \) in terms of the components in both pairs of dual bases. Check that the expressions agree under change of basis.