Problem Set # 10
Multivariable Analysis (M375T)
Due: March 28

1. Recall from lecture that a smooth curve in an affine space $A$ is defined as a subset $C \subset A$ such that there exists a smooth 1:1 immersion $\gamma: I \to A$ with $\gamma(I) = C$, where $I \subset \mathbb{R}$ is an interval. (An immersion is a map with injective differential.)

(a) For $p \in C$ define the tangent line $T_pC$ to $C$ at $p$ as a one-dimensional subspace of the vector space $V$ of translations underlying $A$. Note that $\gamma$ is not data in the definition; its existence is a condition. Thus your definition of $T_pC$ must be independent of the choice of $\gamma$.

(b) Prove that $T_pC$ is independent of $p$ if and only if $C$ is a subset of an affine line in $A$.

2. Let $E$ be a Euclidean space over an inner product space $V$ and $C \subset E$ a smooth curve.

(a) Recall the definition of the acceleration as a vector field

$$\alpha: C \to V$$

That is, write the definition and check that it is well-defined. (First define the set of distinguished unit speed parametrizations.)

(b) Prove that $\alpha(p)$ is perpendicular to $T_pC$ for all $p \in C$.

3. (a) Let $W$ be a vector space and $V \subset W$ a subspace. Define the quotient vector space $W/V$. (Hint: An element of $W/V$ is a coset of $V \subset W$, so a subset of vectors of the form $\xi + V$ where $\xi \in W$. Define addition and scalar multiplication using those operations in $W$.)

(b) Now let $M$ be an affine space over $W$. Define the quotient $M/V$ as an affine space over $W/V$. (Hint: An element of $M/V$ is an orbit of the $V$-action on $M$, so a subset of points of the form $p + V$ where $p \in M$.)

4. Let $V$ be a real vector space with an inner product $\langle -, - \rangle$.

(a) Prove the Cauchy-Schwarz inequality: For all $\xi, \xi' \in V$,

$$|\langle \xi, \xi' \rangle| \leq \|\xi\| \|\xi'\|$$

with equality if and only if $\xi, \xi'$ are proportional.
(b) Prove that
\[ \langle \xi, \xi' \rangle \leq \|\xi\| \|\xi'\| \]
with equality if and only if $\xi, \xi'$ are proportional with a nonnegative constant of proportionality.

(c) Prove that
\[ \|\xi + \xi'\| \leq \|\xi\| + \|\xi'\| \]
as a corollary of Cauchy-Schwarz. When does equality hold?

5. Let $U \subset E$ be an open subset of a Euclidean space and $f: U \to \mathbb{R}$ a $C^2$ function. Define
\[
h: U \to \mathbb{R} \\
p \mapsto \langle \text{grad}_p f, \text{grad}_p f \rangle
\]
Characterize the critical points of $h$.

6. Recall from lecture that a Galilean structure on a vector space $W$ is (i) a codimension one subspace $V \subset W$, (ii) an inner product on $V$, and (iii) an inner product on $W/V$. Let $M$ be an affine space over $W$, so a Galilean spacetime. A curve $C \subset M$ is a *worldline* if $T_p C \oplus V = W$ for all $p \in C$. Let $\pi: W \to W/V$ be the projection. A parametrization $\gamma: I \to W$ of $C$ is *distinguished* if $\pi(d\gamma/d\tau)$ has norm one for all $\tau \in I$. Choose an arbitrary parametrization and write the differential equation for a reparametrization to a distinguished parametrization. Sketch a proof that distinguished parametrizations exist.

7. (a) Let $M$ be a Galilean spacetime and $G$ the group of symmetries of $M$. (It is the group of invertible affine transformations $M \to M$ which preserve the Galilean structure.) Prove that $G$ acts transitively on the set of affine worldlines (which are worldlines of inertial observers).

What is the stabilizer group of an affine worldline?

(b) Repeat for a Minkowski spacetime.

8. Let $M$ be a Minkowski spacetime with speed of light $c$ and underlying vector space $W$. Choose a basis $e_0, e_1, \ldots, e_{n-1}$ of $W$ so that
\[
\langle e_0, e_0 \rangle = c^2 \\
\langle e_i, e_i \rangle = -1, \quad i = 1, 2, \ldots, n - 1 \\
\langle e_\mu, e_\nu \rangle = 0, \quad \mu \neq \nu
\]
Let $t, x^1, \ldots, x^{n-1}$ be an associated coordinate system, so $e_0 = \partial/\partial t$, $e_i = \partial/\partial x^i$. Let $\phi: W \to W^*$ be the isomorphism defined by the metric:
\[
\phi(\xi)(\eta) = \langle \xi, \eta \rangle, \quad \xi, \eta \in W.
\]
Suppose
\[ t = t(\tau) \]
\[ x^i = x^i(\tau) \]
is the worldline of a particle of rest mass \( m_0 \), parametrized by a proper time \( \tau \). Compute the energy-momentum, which is \( m_0 \) times \( \phi \) applied to the proper velocity \( d\gamma/d\tau \). Write the result in terms of the speed \( v^2 = \sum_i (dx^i/dt)^2 = \sum_i (dx^i/d\tau)^2 / (dt/d\tau)^2 \). You should find standard formulas in special relativity.