Problem Set # 12  
Multivariable Analysis (M375T)  
Due: April 11

I will endeavor to make the remaining problem sets shorter (in time, if not space) as you need to be focusing on your projects as well. Be sure to see me regularly, even briefly, to update me on your progress.

1. This problem assumes you have familiarity with complex numbers and even better with basic complex analysis. Let \( \mathbb{C} \) be the complex (affine) line\(^1\) with coordinate \( z \). Write \( z = x + iy \) where \( x, y \in \mathbb{R} \) and \( i^2 = -1 \). Recall the complex conjugate \( \bar{z} = x - iy \). The coordinates \( x, y \) identify \( \mathbb{C} \) with the real affine plane \( \mathbb{A}^2 \).
   
   (a) Write \( x, y \) in terms of \( z, \bar{z} \). We use \( z, \bar{z} \) as (complex) coordinates on \( \mathbb{A}^2 \).

   (b) We use complex differential forms, which are linear combinations of \( dx, dy \) with complex coefficients. Express \( dz, d\bar{z} \) in terms of \( dx, dy \). Define the basis \( \partial/\partial z, \partial/\partial \bar{z} \) dual to \( dz, d\bar{z} \) and express it in terms of \( \partial/\partial x, \partial/\partial y \).

   (c) Let \( U \subset \mathbb{C} \) be an open set. Show that a \( C^1 \) function \( f: U \to \mathbb{C} \) is analytic (holomorphic) if and only if \( \partial f / \partial \bar{z} = 0 \).

   (d) Continuing, define the complex 1-form \( \alpha \in \Omega^1(U; \mathbb{C}) \) by

   \[
   \alpha = f(z, \bar{z}) dz.
   \]

   Show that \( d\alpha = 0 \) if and only if \( f \) is analytic.

2. (a) Let \( V \) be a 3-dimensional vector space. Show that every vector in \( \bigwedge^2 V \) has the form \( \xi_1 \wedge \xi_2 \) for some \( \xi_1, \xi_2 \in V \). (By definition any vector is a sum of such products. Here you prove that every vector is in fact a product. Such elements of the exterior algebra are called decomposable. They represent parallelepipeds.)

   (b) If \( \dim V > 3 \) prove there exists a vector in \( \bigwedge^2 V \) which is not decomposable.

3. Let \( V \) be a vector space. Prove that \( \xi_1, \ldots, \xi_n \in V \) are linearly independent if and only if \( \xi_1 \wedge \cdots \wedge \xi_n \) is nonzero.

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\(^1\)It is a complex line: we navigate with a single complex number, just as we navigate on the real line with a single real number. A complex plane requires two complex numbers to locate a point.
4. Let $V$ be an $n$-dimensional vector space.

(a) Show that $\bigwedge^n V$ is one-dimensional. Show that with the usual topology the space of nonzero vectors in $\bigwedge^n V$ has two (path) components.

(b) Define an equivalence relation on ordered bases of $V$ as follows. An ordered basis $e_1, \ldots, e_n$ determines a nonzero element $e_1 \wedge \cdots \wedge e_n \in \bigwedge^n V$. Two bases are deemed equivalent if the corresponding wedge products lie in the same (path) component. Show that this does define an equivalence relation with two equivalence classes.

(c) Let $e_1, \ldots, e_n$ and $f_1, \ldots, f_n$ be ordered bases, and write

$$f_j = A^j_1 e_1, \quad j = 1, \ldots, n$$

for some matrix $(A^j_1)$. Express $f_1 \wedge \cdots \wedge f_n$ in terms of $e_1 \wedge \cdots \wedge e_n$. (Hint: As always, try low values of $n$ first.) How can you tell from the matrix $A$ if the bases are equivalent?

Some terminology: $\text{Det} V = \bigwedge^n V$ is called the determinant line of $V$. A choice of component of $\text{Det} V \setminus \{0\}$ is called an orientation of $V$.

5. Suppose $V$ is a vector space with inner product $\langle -, - \rangle$. Define an induced inner product on $\bigwedge^2 V$. You may want to consider $V$ finite dimensional with orthonormal basis $e_1, \ldots, e_n$. Then what property does the basis $e_1 \wedge e_2, e_1 \wedge e_3, \ldots, e_1 \wedge e_n, e_2 \wedge e_3, \ldots$ of $\bigwedge^2 V$ have? Suppose $\xi_1 \wedge \xi_2$ represents a parallelogram. What is the geometric interpretation of the norm $\|\xi_1 \wedge \xi_2\|$? What about the inner product between two parallelograms?

6. Integrate the differential form

$$\omega = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy$$

over the sphere $x^2 + y^2 + z^2 = R^2$ in $\mathbb{A}^3$, where $R > 0$ is fixed.

7. Four bugs start at time $t = 0$ at the four corners of the square in $\mathbb{A}^2$ with vertices $(0, 0), (1, 0), (1, 1), (0, 1)$. They move with unit speed always heading towards the bug following them in the cyclic order specified in the previous sentence. Compute the trajectory of each bug.

8. This problem gives you theoretical practice with a definition by a universal property. Let $V$ be a real vector space. We define an exterior algebra to be a pair $(E, \iota)$ of a real algebra $E$ and a linear map $\iota: V \to E$ such that

$$\iota(\xi_1) \iota(\xi_2) = -\iota(\xi_2) \iota(\xi_1), \quad \xi_1, \xi_2 \in V,$$
and if $A$ is any real algebra and $L: V \to A$ a linear map such that

$$L(\xi_1)L(\xi_2) = -L(\xi_2)L(\xi_1), \quad \xi_1, \xi_2 \in V,$$

then there is a unique homomorphism of algebras $f: E \to A$ such that $L = f \circ L$. A homomorphism of algebras is a linear map such that $f(e_1e_2) = f(e_1)f(e_2)$ for all $e_1, e_2 \in E$.

(a) Prove that $(E, \iota)$ obeys a strong uniqueness property: if $(E', \iota')$ is another exterior algebra, then there is a unique isomorphism of algebras $f: E \to E'$ determined by a certain property. Identify that property.

(b) Prove that $\iota$ is necessarily injective. So we can identify $V$ as a subspace of $E$.

(c) One must still prove existence by constructing an example of an exterior algebra. You might do this in case $V$ is finite dimensional by choosing a basis. Or, perhaps, you can do this in general.