Method 1: direct computation

Recall the formula

\[ \text{det} A = C \]

where \( C \) is the determinant of the \((n-1) \times (n-1)\) matrix obtained by deleting the \( i \)th row and \( j \)th column of \( A \).

Then

\[ \frac{\partial \text{det} A}{\partial A_{i,j}^k} = \frac{\partial (\text{det} A)}{\partial A_{i,j}^k} + C_{i,j} \frac{\partial A_{i,j}^k}{\partial A_{i,j}^k} \]

Now observe that \( C_{i,j} \) is computed by deleting the \( i \)th column, while \( A_{i,j}^k \) is in the \( i \)th column. This means \( \frac{\partial C_{i,j}}{\partial A_{i,j}^k} = 0 \). On the other hand, clearly \( \frac{\partial A_{i,j}^k}{\partial A_{i,j}^k} = \delta_{i,j} \), so

\[ \frac{\partial (\text{det} A)}{\partial A_{i,j}^k} = C_{i,j} \]

Applying the chain rule gives

\[ \frac{\partial (\text{det} A)}{\partial t} \bigg|_{t=0} = \frac{\partial (\text{det} A^h)}{\partial t} \bigg|_{t=0} \left. \frac{\partial A_{i,j}^k}{\partial t} \right|_{t=0} = C_{i,j} (0) \left. \frac{\partial A_{i,j}^k}{\partial t} \right|_{t=0} = \text{Tr} C_{i,j} \frac{\partial A_{i,j}^k}{\partial t} (0). \]

In the case \( A (0) \) is invertible, \( C = A^{-1} \text{det} A \), so

\[ \frac{\partial (\text{det} A)}{\partial t} \bigg|_{t=0} = \text{det} A \text{Tr} \frac{\partial A}{\partial t} (0). \]
e) Let $W = \text{span} \{ [T_1, T_2] \}$. We claim

$$W = \text{Ker} \; T_v \subset \text{End}(V)$$

i.e. $T_v \circ W \Rightarrow T_v T = 0$.

It's clear that $W \subset \text{Ker} \; T_v$. Moreover, $T_v : \text{End}(V) \rightarrow \mathbb{R}^n$ is a surjective linear map, so $\dim \text{Ker} \; T_v = n^2 - 1$.

Thus it suffices to produce $n^2 - 1$ linearly ind. elements in $W$.

Fix a basis $\{ e_i \}$ for $V$.

Two computations:

\[
T_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad T_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \Rightarrow [T_1, T_2] = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}
\]

\[
T_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad T_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \Rightarrow [T_1, T_2] = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
\]

Now back to $V$: picking pairs $(i, j)$, construct $A \in W$ s.t.

\[
A(e_i) = e_j \quad A(e_j) = 0 \quad \forall k \neq i
\]

by choosing $T_1, T_2$ as in the first computation when restricted to $\text{span} \{ e_i, e_j \}$. This gives $n^2 - n$ linear transformations.

Then use the second computation to get $n - 1$ $A(e_k) = e_i$. Check that these $n^2 - 1$ elements of $W$ are linearly ind. Their matrices are

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

in $(i, j)$ the place.
7) a) I only prove the triangle inequality, known as Minkowski's inequality.

For $p > 1$, $f: \mathbb{R}^n \to \mathbb{R}$ is convex, meaning that

$$f(\lambda z_1 + (1-\lambda) z_2) \leq \lambda f(z_1) + (1-\lambda) f(z_2) \quad \forall \lambda \in (0,1)$$

we'll use this below. Take $\xi, \eta \in \mathbb{R}^n$, and let $\xi_0 = \frac{\xi}{\|\xi\|_p}$, $\eta_0 = \frac{\eta}{\|\eta\|_p}$.

Let $\lambda = \frac{\|\xi\|_p}{\|\xi\|_p + \|\eta\|_p}$. Then

$$\|\xi + \eta\|_p^p = \sum \lambda |\xi_0^i + \eta_0^i|^p \leq \sum (\lambda |\xi_0^i| + (1-\lambda) |\eta_0^i|)^p$$

$$= \left(\|\xi\|_p + \|\eta\|_p\right)^p \sum (\lambda |\xi_0^i| + (1-\lambda) |\eta_0^i|)^p$$

$$\leq \left(\|\xi\|_p + \|\eta\|_p\right)^p \sum \lambda |\xi_0^i|^p + (1-\lambda) |\eta_0^i|^p$$

Taking $p^{th}$ roots gives

$$\|\xi + \eta\|_p \leq \|\xi\|_p + \|\eta\|_p$$

and likewise for $\eta$. 

The following estimates are used frequently in probability:

\[
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \, dx \leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{\lambda x^2}{2}} \, dx \leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{x} \, dx = \frac{1}{2} e^{-\frac{\lambda}{2}}
\]

\[
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \geq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{x} e^{-\frac{x^2}{2}}
\]

\[
\int_{-\infty}^{\infty} \frac{1}{x} e^{-\frac{x^2}{2}} = \frac{e^{-\frac{\lambda}{2}}}{\lambda} - \int_{-\infty}^{\infty} \frac{1}{x} e^{-\frac{x^2}{2}} \quad \text{(integrate by parts)}
\]

to give

\[
\geq \frac{e^{-\frac{\lambda}{2}}}{\lambda} \frac{1}{\sqrt{8\pi}}.
\]

Substituting \(2\) for \(\lambda\) gives an estimate on the integral in question.