1. In this problem proceed directly from the axioms for a field and a vector space. Let \( V \) be a vector space over a field \( F \). Suppose for some \( c \in F \) and \( \xi \in V \) we have \( c\xi = 0 \). Prove that \( c = 0 \) or \( \xi = 0 \).

2. (a) Prove directly that
\[
\rho_1(\xi) = |\xi^1| + \cdots + |\xi^n| \\
\rho_2(\xi) = \sqrt{(\xi^1)^2 + \cdots + (\xi^n)^2}
\]
are equivalent norms on \( \mathbb{R}^n \). Here \( \xi = (\xi^1, \ldots, \xi^n) \in \mathbb{R}^n \).

(b) Let \( V \) be a vector space with equivalent norms \( \rho_1, \rho_2 \). Prove that a subset \( C \subset V \) is compact in the metric space topology determined by \( \rho_1 \) if and only if it is compact in the metric space topology determined by \( \rho_2 \).

3. Let \( \rho_1, \rho_2 \) be norms on a real vector space \( V \). Prove that \( \rho_1, \rho_2 \) are equivalent if and only if the identity maps
\[
id_V: (V, \rho_1) \rightarrow (V, \rho_2) \\
id_V: (V, \rho_2) \rightarrow (V, \rho_1)
\]
are continuous.

4. Let \( V \) be the vector space of bounded continuous functions \( f: (0, 1) \rightarrow \mathbb{R} \) with the sup norm \( \|f\| = \sup_{x \in (0,1)} |f(x)| \). Consider the sequence \( \{f_n\}_{n=1}^\infty \subset V \) defined by \( f_n(x) = x^n \). Is \( \{f_n\} \) a Cauchy sequence? If so, does it have a limit in \( V \)? What is it?

5. Consider the function \( f: \mathbb{A}^2 \rightarrow \mathbb{R} \) defined by
\[
f(x, y) = \begin{cases} 
\frac{xy}{x^2 + y^2}, & (x, y) \neq (0, 0); \\
0, & (x, y) = (0, 0).
\end{cases}
\]
Is \( f \) continuous at \( (0, 0) \)? Do all directional derivatives exist at \( (0, 0) \)?
6. Give an example of each of the following or prove that no such exists.

(a) A continuous function \( f : (0, 1) \to \mathbb{R} \) which is not differentiable.

(b) A differentiable function \( f : (0, 1) \to \mathbb{R} \) which is not continuous.

(c) A function \( f : \mathbb{R}^2 \setminus \{(0, 0)\} \to \mathbb{R} \) such that

\[
\lim_{y \to 0} \lim_{x \to 0} f(x, y)
\]

exists but

\[
\lim_{(x, y) \to (0, 0)} f(x, y)
\]

does not exist.

7. In this problem you will explore three ways to associate a shape to a function. If \( X, Y \) are sets and \( f : X \to Y \) a function, then

1. the image of \( f \) is the subset \( f(X) \subset Y \),
2. the preimage of \( c \in Y \) is the subset \( f^{-1}(c) \subset X \), and
3. the graph of \( f \) is the subset

\[
\{(x, f(x)) : x \in X\} \subset X \times Y.
\]

Construct the unit circle \( S \subset \mathbb{R}^2 \) in each of these three ways. In other words, find sets \( X, Y \) and functions \( f : X \to Y \) (and for the preimage the point \( c \)) which realize \( S \subset \mathbb{R}^2 \) as an image, preimage, and graph. (Question: Can you realize the entire circle with a single function in each of the three ways, or only a subset of the circle?)

8. Let \( S \) be a metric space and \( W \) a Banach space. Let \( V \) be the vector space of bounded continuous functions \( \xi : S \to W \) with norm

\[
\|\xi\|_V = \sup_{s \in S} \|\xi(s)\|_W.
\]

(a) Prove that \( V \) is a Banach space. (We did several parts of this in lecture.)

(b) Identify the Banach space \( V \) in case \( S \) is a finite set. How does the metric on \( S \) enter?