Problem Set # 5
Multivariable Analysis (M375T)
Due: February 14

1. (a) Let $A$ be an $n \times n$ matrix with real entries. Prove that the infinite series

$$1 + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$

converges. The limit is the exponential $\exp(A) = e^A$.

(b) Compute $e^A$ for

$$A = \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix}$$

(c) Let $M_n$ denote the space of $n \times n$ real matrices. Compute the differential of $\exp: M_n \to M_n$ at the identity matrix.

2. Let $A, B$ be affine spaces modeled on normed vector spaces $V, W$ and $U \subseteq A$ a connected open set. Prove that a differentiable map $f: U \to B$ is the restriction of an affine map $A \to B$ if and only if the differential of $f$ is a constant map $U \to \text{Hom}(V, W)$. (What does this say for $A = B = \mathbb{R}$? Don’t forget to always start with special cases!)

3. Recall that an inner product on a real vector space $V$ is a function $\langle -, - \rangle: V \times V \to \mathbb{R}$ which is linear in each variable when the other is held fixed; $\langle \xi_1, \xi_2 \rangle = \langle \xi_2, \xi_1 \rangle$ for all $\xi_1, \xi_2 \in V$; and for all $\xi \in V$ we have $\langle \xi, \xi \rangle \geq 0$ with equality if and only if $\xi = 0$.

(a) Suppose $\xi_1, \xi_2: (a, b) \to V$ are differentiable curves. Compute $f'(t)$ for $t \in (a, b)$, where $f: (a, b) \to \mathbb{R}$ is defined by

$$f(t) = \langle \xi_1(t), \xi_2(t) \rangle.$$ 

(b) An affine space $E$ over an inner product space $V$ is called a Euclidean space. Suppose $\gamma(t): (a, b) \to E$ is a twice differentiable parametrized curve. Show that $\gamma$ has constant speed if and only if the velocity is orthogonal to the acceleration.

(c) Continuing with the curve $\gamma$, which we view as a motion in $E$ parametrized by the time interval $(a, b)$, what can you say about the motion of the affine approximation to $\gamma$ at $t_0 \in (a, b)$?

(d) Assume $\gamma$ is a unit speed motion. What can you say about the motion defined by the second order Taylor series? You may assume that $E$ is 2-dimensional if you like.
4. Let $X$ be a metric space. We prove in lecture that a *contracting* map $X \to X$ has a fixed point. We can ask if there exists any continuous map $X \to X$ with no fixed point. Find examples of metric spaces for which the answer to that question is ‘yes’ and metric spaces for which the answer to that question is ‘no’.

5. Set up an iterative procedure to compute $x \in [0, 1]$ such that $1 - x^2 = x$.

6. (a) Consider the function

$$f : \mathbb{A}^2 \longrightarrow \mathbb{R}$$

$$(x, y) \longmapsto x^2 + y^2$$

Let $S \subset \mathbb{A}^2$ be $S = f^{-1}(1)$. Fix $(x_0, y_0) \in S$ and compute the affine approximation $\alpha$ to $f$ at $(x_0, y_0)$. What is $\alpha^{-1}(1)$?

(b) Let $M_n$ denote the space of real $n \times n$ matrices, and consider the function

$$f : M_n \longrightarrow M_n$$

$$A \longmapsto AA^t,$$

where $A^t$ is the transpose matrix to $A$. What is $f^{-1}(I) \subset M_n$, where $I$ is the identity matrix? Compute $\alpha^{-1}(I)$, where $\alpha$ is the affine approximation to $f$ at $A = I$.

7. In lecture we ran out of time computing the variation of the length function. Here’s your chance to complete the computation, but in a better notation! Let $W$ be an inner product space (you may assume $W$ is 2-dimensional if you like) with inner product $\langle -, - \rangle$. Let $E$ be an affine space over $W$. Let $A$ denote the affine space of functions $\gamma : [0, 1] \to E$ which are twice differentiable, and let $U \subset A$ be the open set of such functions with nonvanishing velocity: $\dot{\gamma}(t) \neq 0$ for all $t \in [0, 1]$. Define the length $L : U \to \mathbb{R}$ by

$$L(\gamma) = \int_0^1 dt \, \langle \ddot{\gamma}(t), \ddot{\gamma}(t) \rangle^{1/2}$$

Fix $\gamma$ which has unit speed, i.e., $\langle \dot{\gamma}(t), \ddot{\gamma}(t) \rangle = 1$ for all $t \in [0, 1]$, so that $L(\gamma) = 1$. Let $\xi : [0, 1] \to W$ be any twice differentiable function. Compute the directional derivative

$$(\xi L)(\gamma) = \frac{d}{ds} \bigg|_{s=0} L(\gamma + s\xi)$$

You may differentiate under the integral sign without justifying that step. Experiment with your answer to see if it makes sense. Ask and answer questions, such as: What happens if $\xi$ is constant? If $\xi$ vanishes at the initial and final times? What are the critical points of $L$?