# M375 T: Problem Set #5

## Solutions to Selected Problems

### #1

From problem set 4, you know that if

\[ \|A\|_1 = \sup_{v \neq 0} \frac{\|Av\|_2}{\|v\|_2} \]  where \( \|\cdot\|_2 \) is the Euclidean norm on \( \mathbb{R}^n \),

then \( (M_n, \|\cdot\|_1) \) is a Banach space and

\[ \|AB\|_1 \leq \|A\| \|B\|_1. \]  (this is called a Banach algebra).

Then if \( S_m = \sum_{i=m}^{\infty} \frac{A^i}{i!} \), and \( m \geq N \), then

\[ \|S_m - S_{m'}\| \leq \sum_{i=m'}^{\infty} \frac{\|A^i\|}{i!} \leq \sum_{i=m'}^{\infty} \frac{\|A^i\|}{i!} \to 0 \]  as \( N \to \infty \),

since \( \sum_{i=0}^{\infty} \frac{\|A^i\|}{i!} = e^\|A\| < \infty \).

Thus \( \{S_m\} \) is Cauchy, and so has a limit \( \exp(A) \in M_n \), with \( \|S_m - \exp(A)\| \to 0 \).

### #2

You can check that

\[ \exp(A) = \begin{pmatrix} \cosh 2 & \frac{1}{2} \sinh 2 \\ 2 \sinh 2 & \cosh 2 \end{pmatrix} \]

### #3

Assume \( d_1 \exp : M_n \to M_n \) exists. Then for \( B \in M_n \),

\[ d_1 \exp (B) = \lim_{x \to 0} \frac{\exp(I + xB) - \exp I}{x} \]

We know that \( \exp(I + xB) = \sum_{i=0}^{\infty} \frac{(I + xB)^i}{i!} = \sum_{i=0}^{\infty} \frac{1}{i!} \sum_{j=0}^{i} \binom{i}{j} x^j B^j \)
For \( |x| < \frac{1}{2\|B\|} \), \( \frac{1}{2} \sum_{i=2}^{\infty} \|x_iB_i\| \leq x^2 \|B\|^2 \sum_{i=2}^{\infty} x_iB_i^{-2} = x^2 \frac{\|B\|^2}{1 - x\|B\|} \leq x^2 \cdot 2\|B\|^2 

It follows that

\[
\| \exp(I + xB) - eI - xB \| \leq C x^2 \quad (C = \exp(2 \|B\|) \text{ works})
\]

\[
\Rightarrow \lim_{x \to 0} \frac{\exp I + xB - \exp I}{x} = eB, \quad \text{so}
\]

\[
d_1 \exp(B) = eB.
\]

Note: the same argument can be used to show \( d_1 \exp \) exists.

\#4

**Trivial examples:**

\[ \{a\} \quad \text{every map is } f(a) = a, \text{ which has a fixed pt} \]

\[ \{a, b\} \quad f(a) = b, f(b) = a \text{ has no fixed pt.} \]

(with the unique metric)

**Easy examples:**

\( X = [0, 1] \) with usual metric

any map \( f : X \to X \) has a fixed pt, since

\( (f - x)(0) \geq 0, \ (f - x)(1) \leq 0, \) and so by intermediate value theorem \( f \) \( \exists x \) \( f(x) = x, \) \( f(x) - x = 0. \)

\( X \) disconnected always has maps with no fixed pts, for if

\( X = U_1 \cup U_2 \text{ open, } f(U_1) = \{x_1\} \)

\( \text{disjoint union} \)

\( f(U_2) = \{x_1\} \text{ is continuous.} \)

**Hard examples:**

\( X = \overline{B}(r) \subset \mathbb{R}^n \) the closed unit ball.

\( \text{every map } X \to X \text{ has a fixed pt (Brouwer).} \)
Here is one of many possible solutions:

First, compute that

\( 1 - x^2 = x \quad \iff \quad \frac{-1}{3} - \frac{1}{3} (x - 1)^2 = x - 1 \)

Let \( z = x - 1 \). Then a solution of (4) with \( x \in [0, 1] \)

\( \iff \) a fixed point of \( f(z) = \frac{-1}{3} - \frac{1}{3} z^2 \) in \( x = [z, 0] \).

But \( f : X \to X \) is continuous, and moreover

\[ |f(z) - f(z')| \leq \frac{1}{3} |z - z'||(1 + |z| + |z'|) \leq \frac{2}{3} |z - z'|, \]

so \( f \) is a contraction. By the contraction mapping principle, \( f \) will have a fixed point \( x \) that fixed pt. is unique, and for any \( z \in X \), \( f^n(z) \to x \) as \( n \to \infty \).

\( n \)-composition of \( f \).

It follows that \( 1 - x^2 = x \) has a unique solution \( x \) in \([0, 1]\) and

\[ x = \lim_{n \to \infty} f^n(x - 1) + 1 \quad \text{for any } x \in [0, 1]. \]

\#7

\[ L(x_0 \xi) = \int_0^1 \langle \dot{x} + a \xi, \dot{x} + a \xi \rangle^{1/2} dt \]

\[ \frac{d}{da} L(x_0 \xi) = \int_0^1 2 \langle \dot{x} + a \xi, \dot{x} + a \xi \rangle \frac{d}{da} \langle \dot{x} + a \xi, \dot{x} + a \xi \rangle^{1/2} dt \]  
\([\text{We used differentiation under } \int \text{ sign, chain rule, problem 3a, and symmetry of } \langle \cdot, \cdot \rangle \text{ of } \langle \cdot, \cdot \rangle] \]

\[ \implies \frac{d}{da} \bigg|_{a=0} L(x_0 \xi) = \int_0^1 2 \langle \dot{x}, \dot{x} \rangle dt \]  
(here use that \( \langle \dot{x}, \dot{x} \rangle = 1 \))

- If \( \xi \) is constant (i.e. \( \xi = 0 \)) \( \frac{d}{da} \bigg|_{a=0} L(x_0 \xi) = 0 \), which means the length of a curve is invariant under translation.
- If \( \{0\} = \{1\} = 0 \), we can integrate by parts:

\[
\mathcal{L}(\tilde{\gamma}) = \int_0^1 \mathcal{K}(\tilde{\gamma}, \dot{\tilde{\gamma}}) \, dt = \int_0^1 \frac{d}{dt} \langle \dot{\tilde{\gamma}}, \{\tilde{\gamma}\} \rangle - \langle \ddot{\tilde{\gamma}}, \{\tilde{\gamma}\} \rangle \, dt \quad \text{(equating both sides)}
\]

\[
= \langle \dot{\tilde{\gamma}}, \{\tilde{\gamma}(1)\} - \{\tilde{\gamma}(0)\} \rangle - \int_0^1 \langle \ddot{\tilde{\gamma}}, \{\tilde{\gamma}\} \rangle \, dt
\]

by assumption.

This implies that if \( \{\tilde{\gamma}\} \) is orthogonal to the acceleration \( \ddot{\tilde{\gamma}} \), \( \mathcal{L} = 0 \).

For example, if \( \{\tilde{\gamma}\} \) is parallel to \( \dot{\tilde{\gamma}} \), by ph. 4.16 \( \langle \ddot{\tilde{\gamma}}, \{\tilde{\gamma}\} \rangle = 0 \). Thus, any reparametrization of \( \tilde{\gamma} \) will have the same length.

- If \( \mathcal{L}(\tilde{\gamma}) = 0 \) for all \( \{\tilde{\gamma}\} \) with \( \{0\} = \{1\} = 0 \), we have that \( \ddot{\tilde{\gamma}} = 0 \), so \( \tilde{\gamma} \) is a line. This means, for instance, that if for \( x, y \in \mathbb{E} \) fixed we have \( \tilde{\gamma} \) s.t.

\[
\mathcal{L}(\tilde{\gamma}) = \inf \mathcal{L}(\tilde{\gamma}), \quad \text{for} \quad \{\tilde{\gamma}\} \in \mathcal{G}
\]

where \( \mathcal{G} = \{ \{\tilde{\gamma}\} \in \mathbb{E} : \tilde{\gamma}(0) = x, \tilde{\gamma}(1) = y \} \)

and \( \langle \ddot{\tilde{\gamma}}, \dot{\tilde{\gamma}} \rangle \equiv 1 \).

By a previous HW \( \Rightarrow \) \( \mathcal{L}(\tilde{\gamma}) = 0 \) for all \( \{\tilde{\gamma}\} \) with \( \{0\} = \{1\} = 0 \)

\( \Rightarrow \) \( \tilde{\gamma} \) is the line from \( x \) to \( y \).

Such \( \tilde{\gamma} \) are called minimizing geodesics. Note: in our case, this requires \( |x - y| = 1 \).

- If instead \( \mathcal{L}(\tilde{\gamma}) = 0 \) for all \( \{\tilde{\gamma}\} \) (no restrictions on endpoints), in addition to \( \ddot{\tilde{\gamma}} = 0 \) you also have from (i) that \( \dot{\tilde{\gamma}}(0) = 0 \) (choose \( \{\tilde{\gamma}\} \) with \( \dot{\tilde{\gamma}}(1) = 0, \{0\} = \{\tilde{\gamma}\} \)).

But if \( \dot{\tilde{\gamma}}(0) = 0 \) and \( \frac{d}{dt} \langle \dot{\tilde{\gamma}}, \dot{\tilde{\gamma}} \rangle \geq 2, \tilde{\gamma}(1) = 0 \) \( \forall \delta \in [0, 1], \) no \( \tilde{\gamma} \) is constant.

But this contradicts \( \tilde{\gamma} \) having unit speed.