Problem Set # 6
Multivariable Analysis (M375T)
Due: February 21

There are handouts on the website about the chain rule, lagrange multipliers, and the inverse function theorem.

Remember that your first exam is Tuesday, February 19 in class. It will cover everything through the Tuesday, February 12 lecture.

Please read over all the problems before starting. You needn’t tackle the problems in the problem sets in order, and you should definitely work first on those you think are more accessible to you.

1. Consider the function
   \[ f(x, y, z) = x^4 + y^4 + z^4 \]
defined on \( \mathbb{A}^3 \).

   (a) Determine the critical points of \( f \), that is, the points where the differential of \( f \) vanishes.

   (b) Compute the differential of \( f \) in cylindrical coordinates \( r, z, \theta \) given by
   \[
   \begin{align*}
   x &= r \cos \theta \\
y &= r \sin \theta \\
z &= z.
   \end{align*}
   \]

   Do this two ways: 1. Write \( f \) in cylindrical coordinates and then differentiate. 2. Differentiate \( f \) in rectangular coordinates and then change to cylindrical coordinates. Your answers should agree.

   (c) Let \( g: S^2 \to \mathbb{R} \) be the restriction of \( f \) to the unit sphere. What is the maximum value of \( f \)? Where is it attained? (Hint: Compute \( dg \) in whatever coordinate system is most convenient. What is the relationship of \( dg \) to the maximum points?) Can you do a complete analysis of the critical points, i.e., determine the maxima, minima, and saddle points? How many critical points are there? How many critical values (values of \( g \) at the critical points)?

2. (a) Consider the function \( f: \mathbb{R} \to \mathbb{R} \) defined by
   \[ f(x) = \begin{cases} 
   e^{-1/x^2}, & x > 0; \\
   0, & x \leq 0.
   \end{cases} \]

   Prove that \( f \) is \( C^\infty \), i.e., has derivatives of all orders. Sketch the graph of \( f \). Compare \( f \) to its Taylor series at \( x = 0 \).
(b) Given real numbers \( a < b \) show that
\[
g(x) := f(x - a)f(b - x)
\]
is smooth and vanishes outside the interval \((a, b)\).

3. Let
\[
A = \{ f : [0, 1] \to \mathbb{R} : f \text{ is continuous and } f(0) = 1 \}.
\]
Show that \( A \) is an affine space. What is the associated vector space of translations? Show that the sup norm is complete, so \((A, d)\) is a complete metric space for the associated distance function \( d \). (What is \( d \)? You can use previous homeworks to prove the completeness.) Consider \( \phi : A \to A \) defined by
\[
\phi(f)(x) = \int_0^x f(t) \, dt + 1, \quad f \in A.
\]
Is \( \phi \) a contraction? Try to find a fixed point of \( \phi \) by iteration. What happens when you seed the iteration with the constant function \( f \equiv 1 \)? What about other seeds?

4. (a) Show that the map
\[
\mathbb{A}^3 \to \mathbb{A}^3
\]
\[
(x^1, x^2, x^3) \mapsto (\sin x^1, \cos x^2, e^{x^3})
\]
is locally invertible about \((0, \pi/2, 0)\). Is it globally invertible?

(b) Consider
\[
f : \mathbb{A}^2 \to \mathbb{A}^2
\]
\[
(x, y) \mapsto (x^3, y^3)
\]
Is \( df_{(0,0)} \) invertible? Is \( f \) locally invertible about \((0,0)\)? Is \( f \) globally invertible? If there is a local inverse, can it be differentiable at \((0,0)\)?

5. (a) Prove the inverse function theorem for real-valued functions of one variable as follows. Assume \( f : (a, b) \to \mathbb{R} \) is \( C^1 \), so is differentiable on \((a, b)\) and \( f' : (a, b) \to \mathbb{R} \) is continuous. Suppose \( f'(x_0) \neq 0 \) for some \( x_0 \in (a, b) \); say \( f'(x_0) > 0 \). Prove \( f \) is strictly increasing on some interval about \( x_0 \) and use that to construct a local inverse function. Prove that the local inverse is \( C^1 \).

(b) Discuss the inverse function theorem at the origin for the function \( f : \mathbb{R} \to \mathbb{R} \) defined by
\[
f(t) = \begin{cases} 
  t + 2t^2 \sin(1/t), & t \neq 0; \\
  0, & t = 0.
\end{cases}
\]