Problem Set # 7
Multivariable Analysis (M375T)
Due: February 28

There are some computational problems on this problem set. Be sure you can do them! Organize your computations neatly.

1. In $\mathbb{A}^3$ with coordinates $x, y, z$ find the distance from the point $(1, 2, 3)$ to the cone $C = \{(x, y, z) \in \mathbb{A}^3 : x^2 + y^2 = z^2 \}$.

2. A theorem in classical Euclidean geometry, named after the great Napolean, goes as follows. Suppose $A, B, C$ are points in $\mathbb{A}^2$. Construct a point $C'$ external to the triangle $ABC$ so that the triangle $ABC'$ is equilateral. Similarly, construct equilateral triangles $A'BC$ and $AB'C$. Let $A'', B'', C''$ be the centers of the equilateral triangles $A'BC$, $AB'C$, $ABC'$, respectively. The theorem states that $A''B''C''$ is equilateral. You might have fun trying to prove that if you haven’t seen it before—there is a very elegant argument for it. What I’d like you to do is prove that the length of the side of $A''B''C''$ is a smooth function of the points $A, B, C$.

3. Show that there exists $\epsilon > 0$ such that the differential equation

$$f'(x) = f(x) + af(x)^3$$

has a continuously differentiable solution $f : [0, 1] \to \mathbb{R}$ for $|a| < \epsilon$. (Hint: You need to set up Banach spaces to which you’ll apply the implicit function theorem. For the domain you should use the vector space of continuously differentiable ($C^1$) functions $f : [0, 1] \to \mathbb{R}$ with norm

$$\|f\|_{C^1} = \|f'\|_{C^0} + \|f\|_{C^0},$$

where the $C^0$ norm is the sup norm. It would be nice if you prove the completeness. Incidentally, the given differential equation has a solution for all $a$, which follows from a general solvability theorem for ordinary differential equations.)

4. Let $A, B$ be affine spaces with translations normed vector spaces $V, W$, and suppose $f : U \to B$ is a $C^2$ function defined on an open set $U \subset A$. Let $\xi, \eta : U \to V$ be $C^1$ vector fields on $U$.

(a) Define the directional derivative $\xi f : U \to W$. Previously we only defined it for $\xi \in V$, which may be interpreted as a constant vector field $\xi : U \to V$. 
(b) Take $A = \mathbb{A}^n$ to be standard affine space with coordinates $x^1, \ldots, x^n$ and write

$$\xi = \xi^i \frac{\partial}{\partial x^i},$$

$$\eta = \eta^j \frac{\partial}{\partial x^j},$$

where $\xi^i, \eta^j : U \to \mathbb{R}$ are $C^1$ functions. Compute

$$\xi \eta f - \eta \xi f,$$

which is a difference of second derivatives.

5. Continuing #3 on Problem Set #6, prove that $\phi \circ \phi$ is a contraction, and conclude that $\phi$ has a unique fixed point.

6. (a) Consider the functions $f : \mathbb{A}^2 \to \mathbb{A}^3$ and $g : \mathbb{A}^3 \to \mathbb{R}$ defined by

$$f(u, v) = (u + v, uv, u - v)$$

$$g(x, y, z) = x^2 + y + z^2$$

Compute the differential of $g \circ f$ in terms of the variables $u, v$.

(b) Find the critical points of $g \circ f$. What type of critical point are they?

7. Consider the function $f : \mathbb{A}^2 \to \mathbb{R}$ defined by the formula

$$f(x, y) = \cos(x^2 + y^2).$$

(a) Compute the differential of $f$. Your answer should be a linear combination of $dx$ and $dy$ with coefficients functions of $x, y$.

(b) Compute the directional derivative of $f$ at $(1, 1)$ in the direction $(2, 3)$.

(c) Compute the tangent plane to the graph of $f$ at $(\sqrt{\pi}, 0)$.

(d) Compute the tangent line to the level curve $\{ (x, y) : f(x, y) = \frac{1}{2} \}$ at the point $(\sqrt{\pi/6}, \sqrt{\pi/6})$.

(e) Sketch the level curve $\{ (x, y) : f(x, y) = \frac{1}{2} \}$.

(f) What type of critical point is the origin?
8. Spherical coordinates \( r, \phi, \theta \) are related to rectangular coordinates \( x, y, z \) on \( \mathbb{R}^3 \) via the equations

\[
\begin{align*}
x &= \rho \sin \phi \cos \theta \\
y &= \rho \sin \phi \sin \theta \\
z &= \rho \cos \phi.
\end{align*}
\]

(a) Actually this coordinate change is only valid on a subset of \( \mathbb{R}^3 \). Make a precise statement.

(b) Compute equations relating \( \partial/\partial x, \partial/\partial y, \partial/\partial z \) to \( \partial/\partial \rho, \partial/\partial \phi, \partial/\partial \theta \).

(c) Define a multiplication law \( \wedge \) informally on differentials of functions by requiring

\[
df \wedge dg = -dg \wedge df
\]

for any two functions \( f, g \). Now compute \( dx \wedge dy \wedge dz \) in terms of \( d\rho \wedge d\phi \wedge d\theta \). Do you recognize the answer? (Try the analogous problem for polar coordinates in the plane.)