Problem Set # 8

Multivariable Analysis (M375T)

Due: March 7

The first problem is an application of fundamental theorems proved in lecture about the differential of a function. Write a very careful proof and hand it in separately—I will read it to see how your proof-writing skills are developing. State the theorems you use before writing the proof. Map out the steps thoroughly before writing. There are many little steps and I want to see that you have thought through each one.

1. Prove the following theorem. Let \( V \) be a finite dimensional normed vector space; \( A \) an affine space over \( V \); \( W \) a normed vector space; \( B \) an affine space over \( W \); \( U \subset A \) an open set; \( f: U \to B \) a function; and \( p \in U \) a point. Assume that for every \( \xi_1, \xi_2 \in V \) the second directional derivative \( \xi_1 \xi_2 f \) exists on \( U \) and is continuous at \( p \). Then \( df \) is differentiable at \( p \), i.e., the second differential \( d^2f_p \) exists.

2. We use the same notation as in the previous problem.

   (a) Assume \( f \) is a \( C^k \) function, which means that the differential \( d(d(d(\cdots(df)))) \) exists and is continuous (\( k^{th} \) differential). Prove (using mathematical induction) that the \( k^{th} \) differential at a point \( p \) is a symmetric multilinear map

   \[
d^k f_p : V \times V \times \cdots \times V \to W
   \]

   (b) Show that if \( f \) is \( C^k \) then the iterated directional derivative \( \xi_1 \xi_2 \cdots \xi_k f \) exists and is independent of the ordering of the \( k \) vectors \( \xi_1, \xi_2, \ldots, \xi_k \in V \).

3. We continue with the same (usual) notation. Suppose in addition that \( V \) has an inner product \( \langle -, - \rangle \) (see Homework #5, problem #3). Recall that the inner product determines a norm \( \| \xi \| = \sqrt{\langle \xi, \xi \rangle} \). It is a fact that this norm is complete if and only if the inner product satisfies the following property: for each bounded linear functional \( \lambda: V \to \mathbb{R} \) there exists a unique \( \xi_\lambda \in V \) such that

   \[
   \lambda(\xi) = \langle \xi_\lambda, \xi \rangle, \quad \xi \in V.
   \]

   In that case we call \((V, \langle -, - \rangle)\) a Hilbert space. Any finite dimensional inner product space is complete (why?).
(a) Define the gradient $\nabla f_p \in V$ as the unique vector which satisfies

$$df_p(\xi) = \langle \nabla f_p, \xi \rangle, \quad \xi \in V.$$ 

Prove that if $p$ is not a critical point, then $\nabla f_p$ points in the direction of maximal increase of $p$ at $p$.

(b) Fix $p_0 \in A$ and set $f(p) = d(p, p_0)^2$. Compute $\nabla f$, a vector field on $A$.

(c) Let $f = f(r, \theta)$ be a function on the plane (rather, an open subset of the plane on which $r, \theta$ are well-defined coordinates). Compute $\nabla f$ in terms of partial derivatives in polar coordinates. Use the standard metric on the plane.

4. Suppose $T: V \rightarrow W$ is a bounded linear map between normed linear spaces. Prove that $T$ is differentiable and $dT = T$.

5. Let $V$ be a finite dimensional inner product space and $B: V \times V \rightarrow \mathbb{R}$ a symmetric bilinear form. Let $S \subset V$ be the sphere of unit norm vectors. Prove that $B$ has an eigenvector by extremizing the function

$$f: S \rightarrow \mathbb{R}$$

$$\xi \mapsto B(\xi, \xi)$$

By restricting to the orthogonal complement (define!) of the line spanned by the eigenvector you found, continue by induction to diagonalize $B$.

6. Express the system of differential equations

$$\frac{dx}{dt} = t + x^2 + y^3$$

$$\frac{dy}{dt} = \cos(xy)$$

as a problem to find an integral curve of a time-varying vector field $\xi(t)$. On what space is the vector field? Is $\xi(t)$ uniformly Lipschitz for each $t$? Is it locally uniformly Lipschitz?

7. The iteration for finding a solution $f(t)$ to an ODE with $f(t_0) = p_0$ is

$$f_n(t) = p_0 + \int_{t_0}^{t} ds F(s, f_{n-1}(s)).$$

Compute the first 3 steps of the iteration for the differential equation

$$\frac{dx}{dt} = t + x$$

beginning with the zero function. Can you guess a solution? Experiment with other starting points for the iteration.
8. Let

\[ F, G : I \times \mathbb{R}^4 \rightarrow \mathbb{R} \]

be smooth functions, where \( I \subseteq \mathbb{R} \) is an open interval. Formulate the problem of solving the system

\[
\begin{align*}
    f''(t) &= F(t, f(t), g(t), f'(t), g'(t)) \\
    g''(t) &= G(t, f(t), g(t), f'(t), g'(t))
\end{align*}
\]

of ordinary differential equations for functions \( f, g : I \rightarrow \mathbb{R} \) as the problem of finding integral curves of a time-varying vector field.