Test # 2
Multivariable Analysis (M375T)
April 18, 2013

True/False. _____/30  1. _____/25  2. _____/20  3. _____/25

Total Score. _____/100  4 (EC). _____/10  5 (EC). _____/10  6 (EC). _____/10

Put your name at the top of the exam. Please explain your work thoroughly and write neatly. Please use scratch paper for your scratch work and present coherent solutions. If you need more space, continue your work on the backs of pages or on the extra sheets in the back. If your solution runs over onto these pages, please indicate that clearly. Staple scratch pages to your exam.

Read through the entire exam first. Only attempt the extra credit problems after completing the rest of the exam. The "Total Score" does not include the extra credit problems.

Please ask if there is any notation you do not understand.

Good luck!

Answer the following true/false questions. If you find a question ambiguous, write a brief explanation. Each true/false question is worth 3 points.

True  False

○  ○  1. The map

\[ x = s^2 t \cos(\pi s t) \]
\[ y = s t^2 \sin(\pi s t) \]

from \( \mathbb{A}^2 \) to \( \mathbb{A}^2 \) has a local inverse mapping a neighborhood of \( (x, y) = (-1, 0) \) to a neighborhood of \( (s, t) = (1, 1) \).

○  ○  2. The differential equation \( x'(t) = x(t) \sin(x(t)) \) has a unique solution \( x: \mathbb{R} \to \mathbb{R} \) with \( x(0) = 0 \).

○  ○  3. Let \( C \subset E \) be a smooth curve in a Euclidean space and \( \alpha: C \to V \) the acceleration vector field relative to a unit speed parametrization. (\( V \) is the vector space of translations in \( E \).) Then \( \alpha \) has unit length.

○  ○  4. If \( f: U \to \mathbb{R} \) is a \( C^2 \) function, where \( U \subset \mathbb{A}^2 \) is an open set, and we use standard coordinates \( x^1, x^2 \) on \( \mathbb{A}^2 \), then

\[ \frac{\partial^2 f}{\partial x^2 \partial x^1} = \frac{\partial^2 f}{\partial x^1 \partial x^2} \]

on \( U \).
5. Suppose $U \subset \mathbb{A}^n$ is an open subset and $\alpha \in \Omega^1(U)$. Let $C_1, C_2 \subset U$ be the image of 1:1 immersions $\gamma_1, \gamma_2 : [0, 1] \to U$ with $\gamma_1(0) = \gamma_2(0)$ and $\gamma_1(1) = \gamma_2(1)$. Then

$$\int_{C_1} \alpha = \int_{C_2} \alpha$$

6. Let $\mathbb{E}^n$ be $n$-dimensional Euclidean space and $f : U \to \mathbb{R}$ a $C^1$ function defined on an open set $U \subset \mathbb{E}^n$. Let $\nabla f_p$ denote the gradient of $f$ at $p \in U$. Then the directional derivative of $f$ in the direction $\nabla f_p$ at $p$ is positive.

7. Let $V$ be a vector space and $\xi_1, \ldots, \xi_k \in V$. Then $\xi_1 \wedge \cdots \wedge \xi_k \in \wedge^k V$ vanishes if and only if there is a linear relation among the vectors $\xi_1, \ldots, \xi_k$.

8. The differential equation $x'(t) = x(t)^2$ has a solution $x : \mathbb{R} \to \mathbb{R}$ with $x(0) = 1$.

9. Let $V$ be a 4-dimensional real vector space and $L \subset V$ a 1-dimensional subspace. Suppose $V$ is oriented and $e_1, e_2, e_3, e_4$ is an oriented basis. Then the basis $e_3, e_4, e_2, e_1$ is also oriented.

10. There exists a $C^1$ function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x + 1) = f(x)$ and $f$ has no critical points.
1. (25 points)

(a) State the implicit function theorem carefully.

(b) Let \( S \subseteq \mathbb{A}^4 \) be the set of solutions \((x, y, z, w) \in \mathbb{A}^4\) to the equations

\[
x^2 + y^2 + z^2 + w^2 = 4 \\
x^3 + 2y^4 + z^5 = 4
\]

Prove that a neighborhood \( U \) of \((1, 1, 1, 1) \in S\) is the graph of two functions

\[
z = z(x, y) \\
w = w(x, y)
\]

of two variables.

(a) In a general form: let \( A, B, C \) be affine spaces over Banach spaces \( V_1, W_1, X \). Let \( U_1, U_2 \) be open sets in \( A, B \); \( F: U_1 \times U_2 \to C \) a \( C^1 \) function; and \((p, q) \in U_1 \times U_2\) be the second partial differential \( dF_{(p, q)}: W \to X \) is an isomorphism. Then there exist open sets \( U_1' \subseteq U_1 \), \( U_2' \subseteq U_2 \) and a \( C^1 \) function \( g: U_1' \to U_2' \) so that \( F(x, g(x)) = c \) for all \( x \in U_1' \), \( g(p) = q \), where \( F(p, q) = c \).

(b) Define \( F: \mathbb{A}^4 \to \mathbb{A}^2 \) by

\[
F(x, y, z, w) = (x^2 + y^2 + z^2 + w^2, x^3 + 2y^4 + z^5)
\]

Write \( \mathbb{A}^4_{x, y, z, w} = \mathbb{A}^2_x \times \mathbb{A}^2_z \). Then \( dF_{(1, 1, 1)} \) is represented by the matrix

\[
\begin{pmatrix}
2x & 2w \\
5z^4 & 0
\end{pmatrix}
\]

which is invertible. So the implicit function theorem applies.
2. (20 points) Give an example of each of the following. You needn’t prove your answers, but do give a justification.

(a) A vector space $V$ and an element $\pi \in \bigwedge^2 V$ such that $\pi$ is not decomposable, i.e., not of the form $\xi_1 \wedge \xi_2$ for some $\xi_1, \xi_2 \in V$.

(b) A smooth curve $C \subset \mathbb{A}^3$ and a 1-form $\alpha$ defined in an open neighborhood of $C$ so that $\int_C \alpha$ is nonzero.

(c) A $C^1$ function which is not $C^2$. Please specify carefully the domain and codomain of your function.

(a) $V = \mathbb{R}^4$ with basis $e_1, e_2, e_3, e_4$. Let $\pi = e_1 e_2 + e_3 e_4$. Note $\pi \wedge \pi = 2 e_1 e_2 e_3 e_4 \neq 0$, but if $\pi = \frac{5}{3} 1 \frac{5}{3}$ then $\pi \wedge \pi = 0$.

(b) $C = \{ (x, 0, 0) : 0 \leq x \leq 1 \} \subset \mathbb{A}^3$ and $\alpha = dx$. Then $\int_C x = 1$.

(c) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \frac{1}{2} x \left| x \right|$. Then $f'(x) = \left| x \right|$ is continuous but not differentiable.
3. (25 points) Consider the functions \( f, g: \mathbb{A}^2 \to \mathbb{R} \) defined by the formula

\[
\begin{align*}
f(x, y) &= \exp(x^2 - y^2) \\
g(x, y) &= xy.
\end{align*}
\]

(a) Compute the first and second differentials of \( f \). Be sure to say carefully what kinds of objects these are.

(b) What type of critical point does \( f \) have at the origin?

(c) Compute \( df \wedge dg \).

(d) Does there exist a function \( h: \mathbb{A}^2 \to \mathbb{R} \) such that \( dh = f dg \)?

(e) Let

\[
\varphi: \mathbb{A}^2 \to \mathbb{A}^2 \\
(u, v) \mapsto (u + v, uv)
\]

Compute \( \varphi^*(f dg) \).

\[
\begin{align*}
\text{(a) } df &= \exp(x^2 - y^2) \left( 2xdx - 2ydy \right) \\
\text{d}^2 f &= A^2 \to \mathbb{R}^2\times\mathbb{R}^2 \\
\begin{pmatrix}
4x^2 + 2 & -4xy \\
-4xy & 4y^2 - 2
\end{pmatrix}
\end{align*}
\]

\[
\begin{align*}
\text{(b) } f \text{ has a saddle point.}
\end{align*}
\]

\[
\begin{align*}
\text{(c) } dg &= ydx + xdy \\
\text{d}^2 df dg &= \exp(x^2 - y^2) (2y^2 + 2x^2) dx \wedge dy
\end{align*}
\]

\[
\begin{align*}
\text{(d) If so, then } \Delta h &= \text{d}^2 f dg, \text{ which is false. So: NO!}
\end{align*}
\]

\[
\begin{align*}
\text{(e) } x &= u + v \\
y &= uv \\
dx &= du + dv \\
dy &= udv + vdu
\end{align*}
\]

\[
\begin{align*}
\text{(e) } 5 \mathbf{f} dg &= \exp(x^2 - y^2) (ydx \wedge xdy) \\
&= \exp\left(\frac{(u + v)^2 - u^2v^2}{2}\right) \left[ udu + udv \right] \\
&= \exp\left(\frac{(u + v)^2 - u^2v^2}{2}\right) \left[ 2udv + v^2du + 2uvdu \right].
\end{align*}
\]
4. (10 points) **(Extra Credit)** Prove that there is a unique solution \( x \in [0.1, 0.9] \) to the polynomial equation \( x^5/5 - x + 0.1 = 0 \).

Let \( f : [0.1, 0.9] \rightarrow [0.1, 0.9] \) be the function \( f(x) = \frac{x^5}{5} + 0.1 \). Note if \( x \in [0.1, 0.9] \), then \( 0.2 \leq x^5 \leq 0.2 \), so \( f(x) \in [0.1, 0.9] \). Since \( |f'(x)| = |x| < 0.9 \), it follows from a basic lemma we used that if \( \|f\| \leq C \), then \( \|f(x) - f(y)\| \leq C \|x - y\| \) with the constant hypothesis. That \( f \) is a contraction. Now \( [0.1, 0.9] \) is a complete metric space — it is closed and bounded by Heine–Borel — so the contraction mapping fixed point theorem applies.

Alternatively, if \( g(x) = \frac{x^5}{5} - x + 0.1 \), then \( g(0.1) > 0 \), \( g(0.9) < 0 \), and \( g'(x) = x^4 - 1 < 0 \) on \( [0.1, 0.9] \). So \( g \) is monotonic decreasing by the intermediate value theorem has a zero, which is then unique.
5. (10 points) (Extra Credit) Exhibit a differential form \( \alpha \in \Omega^1(A^2 \setminus \{0\}) \) such that \( d\alpha = 0 \) and there does not exist \( f \in \Omega^0(A^2 \setminus \{0\}) \) with \( df = \alpha \). Prove these properties.

In polar coordinates \( \Theta \) is not a globally defined function, but \( d\Theta \) is a well-defined form away from the origin.

Use the formulas

\[
\begin{align*}
  x &= r \cos \Theta \\
y &= r \sin \Theta \\
dx &= \cos \Theta \, dr - r \sin \Theta \, d\Theta \\
dy &= \sin \Theta \, dr + r \cos \Theta \, d\Theta
\end{align*}
\]

to find

\[
\begin{align*}
r \, d\Theta &= \cos \Theta \, dy - \sin \Theta \, dx \\
r^2 \, d\Theta &= r \cos \Theta \, dy - r \sin \Theta \, dx \\
&= x \, dy - y \, dx
\end{align*}
\]

Or

\[
\alpha = d\Theta = \frac{x \, dy - y \, dx}{x^2 + y^2}
\]

Now check by direct computation that \( d\alpha = 0 \), or in any small open set \( \Theta \) is well-defined and \( \alpha = d\Theta \), so \( d\alpha = d^2\Theta = 0 \).

If \( C \) is unit circle, oriented counterclockwise, then \( \int_C d\alpha = 2\pi \).

If \( C = \text{unit circle}, \) oriented counterclockwise, then by our "curved fundamental theorem of calculus",

But if \( \alpha = df \), then by our "curved fundamental theorem of calculus",

\[
\int_C df = 0.
\]
6. (10 points) **(Extra Credit)** Let $U \subset \mathbb{A}^2$ be an open set, $g: U \to \mathbb{R}$ a function, and $x^1, x^2: \mathbb{A}^2 \to \mathbb{R}$ the standard coordinate functions. Assume the partial derivatives $\partial g/\partial x^1, \partial g/\partial x^2$ exist and are bounded in $U$. Prove that $g$ is continuous.

Suppose $\left| \frac{\partial g}{\partial x^1} \right|, \left| \frac{\partial g}{\partial x^2} \right| \leq M$ on $U$. Use

The norm $\| (\tilde{x}^1, \tilde{x}^2) \| = |\tilde{x}^1| + |\tilde{x}^2|$ on $\mathbb{R}^2$. Given $\varepsilon > 0$, if

$\| (y^1, y^2) - (x^1, x^2) \| < \frac{\varepsilon}{M}$, then

$$\| g(y^1, y^2) - g(x^1, x^2) \| \leq \| g(y^1, y^2) - g(y^1, x^2) \| + \| g(y^1, x^2) - g(x^1, x^2) \|$$

$$\leq M \| y^2 - x^2 \| + M \| y^1 - x^1 \|$$

$$= M \| (y^1, y^2) - (x^1, x^2) \|$$

$$< \varepsilon.$$

This proves $g$ is **Lipschitz continuous**. (See handout on second derivatives for more info.)

\[
\begin{array}{c}
\text{(\tilde{y}^1, \tilde{y}^2)} \\
\text{(\tilde{x}^1, \tilde{x}^2)}
\end{array}
\]