1) (15 points) Take the 2D vector \( \vec{v} = 2\hat{i} + 3\hat{j} \).
   a) Compute \( \vec{v}^2 \)
   b) Compute \( \vec{v}^* \)
   c) Compute the angle between \( \vec{v}^2 \) and \( \vec{v}^* \).

2) (15 points) Let \( L_1 \) be the line \( 2x - 3y = 3 \) in 2D.
   a) Find the direction of the line.
   b) Find a vector parametric equation for the line through \( P(0, 1) \), parallel to \( L_1 \).

3) (10 points) Find a parametric equation for the curve \( 4x^2 + 9y^2 = 1 \).

4) (20 points) Given the 3D lines \( x = 1 - t, \ y = 2 + t, \ z = 2t \) and \( \vec{r} = (1, 0, -1) + t(1, 1, 1) \), find a vector parametric line through \( P(0, 0, 0) \) that's also perpendicular to both of the other two lines.

5) (30 points) Use vector methods to determine whether any of the following lines in 2D are parallel, perpendicular, or neither; show work!

   \[ L_1 : \ 2x + 4y = 1 \]
   \[ L_2 : \ \vec{r} = (1, 0) + t(1, 2) \]
   \[ L_3 : \ x = 1 - 2y \]
   a) \( L_1 \) and \( L_2 \)
   b) \( L_1 \) and \( L_3 \)
   c) \( L_2 \) and \( L_3 \)

6) (10 points) Use vector methods to find the area of the triangle with vertices at \( P(0, 0), \ Q(1, 1), \ R(2, 4) \)
1) \( \vec{V} = (2, 3) \) \( \vec{V}^* = 45^\circ \text{ c.w. rotation} \)
\[ \frac{1}{\sqrt{2}} (2-3, 2+3) = \frac{1}{\sqrt{2}} (5, -1) \]
\( \vec{V}' = 45^\circ \text{ c.w. rotation} \)
\[ \frac{1}{\sqrt{2}} (2+3, 2-3) = \frac{1}{\sqrt{2}} (5, 1) \]
\( \cos \theta = \frac{\vec{V}_x \cdot \vec{V}^*}{|\vec{V}_x||\vec{V}^*|} \)
\( \theta \) starts with \( \vec{V}' \cdot \vec{V}^* \)
\[ \frac{1}{\sqrt{2}} (-1, 5) \cdot \frac{1}{\sqrt{2}} (5, 1) = \frac{1}{2} \cdot \frac{1}{2} [-5 + 5] = 0 \]
"They are 90° apart!"

2) a) Since \( L \) is \( 2x - 3y = 3 \), \( \vec{n} = (2, -3) \). The direction is \( \perp \) to \( \vec{n} \), so \( \cos \left( \frac{x}{x-3} \right) = -3 \vec{c} - \vec{g} = -(3, 2) \)
\[ \vec{n} = \pm (3, 2) \]

b) This would be \( \vec{F} = \vec{r}_0 + t \vec{V} \) with \( \vec{r}_0 \) = \( P(0, 1) \) and \( \vec{V} = (3, 2) \), so \( \vec{F} = (0, 1) + t(3, 2) \)

3) \( x = \frac{1}{2} \cos t \), \( y = \frac{1}{2} \sin t \) note
\[ 4\left( \frac{\sqrt{2}}{2} \cos t \right)^2 + 9 \left( \frac{1}{3} \sin t \right)^2 = 4 \cdot \frac{1}{4} \cos^2 t + 9 \cdot \frac{1}{9} \sin^2 t = 1 \]
4) I'll take \( \overrightarrow{v} = (-1, 1, 2) \) and \( \overrightarrow{w} = (4, 1, 1) \) as the directions of the two lines, and then the line I want has direction \( \overrightarrow{u} = \pm \overrightarrow{v} \times \overrightarrow{w} \)

\[
\overrightarrow{v} \times \overrightarrow{w} = \det \begin{pmatrix}
\hat{i} & \hat{j} & \hat{k} \\
-1 & 1 & 2 \\
1 & 1 & 1 \\
\end{pmatrix} = \hat{i} \det \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} - \hat{j} \det \begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix} + \hat{k} \det \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \\
= -\hat{i} + 3\hat{j} - 2\hat{k}. \] I'll use \( \overrightarrow{u} = 3\hat{j} - 2\hat{k} \)

Then \( \overrightarrow{F} = \overrightarrow{u} + t\overrightarrow{v} = (0, 3, -2) + \epsilon (1, -3, 2) \)

5) I'll need directions of the lines:

\( L_1: 2x + 4y = 1 \) \( \overrightarrow{n} = (2, 4), \overrightarrow{v_1} = (4, -2) \)

\( L_2: \overrightarrow{v} = \text{coeff of } t \) \( \overrightarrow{v_2} = (1, 2) \)

\( L_3: \text{write c, } 2y + x = 1 \) \( \overrightarrow{n} = (4, 2), \overrightarrow{v_3} = (2, -1) \)

a) \( L_1, L_2 \) check \( \perp = \overrightarrow{v}_1 \cdot \overrightarrow{v}_2 = (4, -2) \cdot (1, 2) = 4 - 4 = 0 \)

\( \rightarrow \text{yes} \) / \( \text{no} \)

5) \( L_2, L_3 \) \( \overrightarrow{v}_2 \cdot \overrightarrow{v}_3 = (1, 2) \cdot (2, -1) = 2 - 2 = 0 \)

\( \rightarrow \text{yes} \) / \( \text{no} \)
1)(15 points) Find an implicit equation for the line through $P(-2, -1)$, perpendicular to the line $\vec{r} = (-1, 2) + s(1, 1)$.

2)(15 points) Let $P(1, 0, 0), Q(0, 1, 0), R(0, 0, 1)$ be vertices of a triangle in 3D. Use vector methods to find the area.

3)(30 points) Let $\vec{v} = (1, -1, 1), \vec{w} = (2, 0, -2), \vec{z} = (-4, 0, 2)$ be vectors. Use vector methods to determine which are parallel, perpendicular. Show work!
   a) $\vec{v}$ and $\vec{w}$
   b) $\vec{v}$ and $\vec{z}$
   c) $\vec{w}$ and $\vec{z}$

4)(20 points) A line in 2D has equation $x = 1 + 2y$.
   a) Find a direction of the line.
   b) Find a vector parametric equation for the line $2y - x = 2$.
   c) Find an implicit equation for a line perpendicular to $2y - x = 2$, through the point $P(0, 1)$

5)(20 points) Let $\vec{v} = (2, 1); \vec{w} = (1, -1)$;
   a) Compute $\vec{v}^\star, \vec{w}^\star$
   b) Check that the cosine of the angle between $\vec{v}$ and $\vec{w}$ is the same as that between $\vec{v}^\star$ and $\vec{w}^\star$
E15 Solutions

1) Slowing down: line is \( \perp \) to \( \vec{v} = (5, 1) \)
   so line has direction \( \vec{w} = (1, -1) \)
   New the normal is \( \perp \) to the direction \( \vec{w} \), so
   The normal is \( (1, 1) \)
   So: \( \vec{n} \cdot (\vec{r} - \vec{r}_0) = (1, 1) \cdot [\begin{pmatrix} x - 2 \\ y - 1 \end{pmatrix}] = 0 \)
   \( x + 2 + y - 1 = 0 \)
   \( x + y = -1 \)

2) Area = \( \frac{1}{2} | \vec{PQ} \times \vec{PR} | \)
   \( \vec{PQ} = \vec{Q} - \vec{P} = (-1, 0, 1) \)
   \( \vec{PR} = \vec{R} - \vec{P} = (-1, 0, 1) \)
   \( \vec{Q} \times \vec{R} = (0, 0, 1) - (0, 0, 1) = (0, 0, 0) \)

2) I'll try first, so use dot product
   a) \( \vec{v} \cdot \vec{w} = (1, -1, 1) \cdot (2, 0, -2) = 2 + 0 - 2 = 0 \)
      \( \perp \) yes \( \parallel \) no

   b) \( \vec{v} \cdot \vec{z} = (1, -1, 1) \cdot (-4, 0, 2) = -4 + 0 + 2 = -2 \neq 0 \)
      Are they parallel? \( \vec{v} \cdot \vec{z} = (1, -1, 1) \)?
      \( o = \frac{1}{2} \)
      \( \perp \) no \( \parallel \) no

   c) \( \vec{w} \cdot \vec{z} = (2, 0, -2) \cdot (-4, 0, 2) = -8 + 0 - 4 = -12 \neq 0 \)
      Parallel? \( (2, 0, -2) = t (-4, 0, 2) \)
      \( -2t = 2 \)
      \( t = -1 \)
      \( \perp \) no \( \parallel \) no
4) \( x = 1 + 2t \), \( y = 1 + 2t \). You could see this as parametric.

\( \text{Equation:} \quad x = 1 + 2t \), \( y = 1 + 2t \)

\( \overrightarrow{v} = (2, 1) \)

5) \( x = 2y + 2 \), \( y = t \), \( x = 2t + 2 \)

\( \overrightarrow{F} = (2t + 2, t) = (2, 0) + t(2, 1) \)

a) \( \overrightarrow{v} = (2, 1) \)

5) \( \text{Rewrite:} \quad x = 2y + 2 \), \( y = t \), \( x = 2t + 2 \)

\( \overrightarrow{F} = (2t + 2, t) = (2, 0) + t(2, 1) \)

c) \( \text{Slow way:} \quad 2y - x = 2 \quad \text{and} \quad x = 2y + 2 \)

This has direction \( \overrightarrow{v} = (2, 1) \).

The \( \perp \) has direction \( \overrightarrow{w} = (1, -2) \).

To find an implicit equation, I need a normal. Which is this \( \overrightarrow{v} \) to \( \overrightarrow{w} \) so \( \overrightarrow{n} = (2, 1) \).

So \( \overrightarrow{n} \cdot (\overrightarrow{v} - \overrightarrow{w}) = 0 \) \( \Rightarrow \overrightarrow{n} \cdot (2, 1) \cdot (y - 0, y - 1) = 0 \)

\[ 2x + y - 1 = 0 \]

5) \( \overrightarrow{V} = (2, 1) \), \( \overrightarrow{w} = (1, 1) \)

\( \overrightarrow{V} = \frac{1}{\sqrt{5}} (2, 1) \), \( \overrightarrow{w} = \frac{1}{\sqrt{2}} (1, 1) \)

\( \overrightarrow{w} = (1, 1) \), \( \overrightarrow{w} = \frac{1}{\sqrt{2}} (1, 1) \)

\( \overrightarrow{W} = \frac{1}{\sqrt{5}} (2, 1) \)

\( \cos \theta = \frac{\overrightarrow{V} \cdot \overrightarrow{W}}{\overrightarrow{V} \cdot |\overrightarrow{W}|} = \frac{(2, 1) \cdot (1, 1)}{\sqrt{5} \sqrt{2}} = \frac{1}{\sqrt{10}} \)

\( \cos \theta = \frac{\sqrt{2} (1, 1) \cdot \sqrt{2} (2, 0)}{\sqrt{2 + 2} \sqrt{4 + 0}} = \frac{\sqrt{2} (1, 1) \cdot \sqrt{2} (2, 0)}{\sqrt{10} \sqrt{2}} = \frac{2}{\sqrt{2}} \)