14U: Falling Bodies

We are going to look at a problem like logistic growth. We drop a ball from a great height, and ask how fast it can go as it falls. So: $v(t) =$ the velocity of the ball. Since the ball is falling, $v(t) \leq 0$. Since we dropped the ball, we have the initial condition $v(0) = 0$.

The velocity of the ball satisfies the diffeq:

$$\frac{dv}{dt} = -g + rv^2$$

Here $g$ is the force of gravity, and $r$ is the resistance to falling caused by air. $g$ pulls the ball down, which is why there’s a negative in front of $g$. Air resistance tends to push the ball back up, which is why there’s a positive. I’m going to rewrite the equation to make it look more like the logistic equation:

$$\frac{dv}{dt} = -g + rv^2 = r(v^2 - \frac{g}{r}) = r(v^2 - K^2) = r(v - K)(v + K)$$

Here, $K$ has to be $K = \sqrt{\frac{g}{r}}$.

1) Show the equilibrium points of the equation are $v = \pm K$.

As we’re interested in $v(t) \leq 0$, the equilibrium we want is $v = -K$. This will turn out to be the terminal velocity. Now, we’ll start with what the diffeq looks like for $v$ near zero. Then $r(v - K)(v + K) \approx r(0 - K)(v + K) = -rK(v + K)$

2) Use the initial condition to show

$$v \approx K(e^{-rKt} - 1)$$

Sketch a graph of this, near $v = 0$

Now we want to see what the equation is like for $v \approx -K$. Then $r(v - K)(v + K) \approx r(-K - K)(v + K) = -2rK(v + K)$

3) Use the initial condition to show

$$v \approx -K(1 - e^{-2rKt})$$

Sketch a graph of this, near $v = -K$. Notice that unlike the logistic equation, there’s no point of inflection in this graph.

4) Using 3), show

$$\lim_{t \to \infty} v(t) = -K$$
So, we have reason to think that \(-K\) is the terminal velocity.

The equation

\[
\frac{dv}{dt} = r(v - K)(v + K)
\]

is separable.

5) Use partial fractions to show that the solution becomes

\[
\frac{1}{2K} \int \left( \frac{1}{v - K} - \frac{1}{v + K} \right) dv = \int rt \, dt
\]

and then

\[
\ln \left| \frac{v - K}{v + k} \right| = 2Krt + C
\]

6) Solve the above for \(v\) remembering that \(v(0) = 0\) and \(v - K > 0\), and get:

\[
v = -K \left( \frac{1 - e^{-2Krt}}{1 + e^{-2Krt}} \right)
\]

7) Use L’Hospital’s Rule to show, finally, that

\[
\lim_{t \to \infty} v(t) = -K
\]

So \(v(\infty) = -K\) is the terminal velocity.

In the logistic equation, we saw there was an inflection point, but the graphs here seem to show \(v(t)\) is concave up always. That means the second derivative is positive.

8) Show

\[
\frac{d^2v}{dt^2} > 0
\]

It’s time to get some numbers in here. Let \(g = 32 \text{ft/sec}^2\). Now we’ll look at different \(r\)’s.

9) a) If you are falling and curl yourself into a ball, \(r = (5.18)10^{-4} \frac{1}{ft}\). What is \(K\), in miles per hour?

9) b) If you are falling but spread yourself out, \(r = (10.4)10^{-4} \frac{1}{ft}\). What is \(K\), in miles per hour?

9) c) If you are falling with a 20-ft parachute, \(r = (13)10^{-2} \frac{1}{ft}\). What is \(K\), in miles per hour?
1) To be at equilibrium means to not change—so it means \( \frac{dv}{dt} = 0 \) or

\[
V(u-k)(u+k) = 0, \quad \text{The factors are zero}
\]

\[V-k = 0 \rightarrow V = k\]

\[V+k = 0 \rightarrow V = -k\]

2) \( \frac{dv}{dt} = r(u-k)(u+k) \approx -rk(u+k) \) so

\[
\frac{dv}{u+k} = -rk \, dt \quad \text{or} \quad \int \frac{dv}{u+k} = -\int rk \, dt
\]

\[\ln|u+k| = -rk \, t + C\]

\[e^{\ln|u+k|} = e^{-rk \, t + C} = e^{-rk \, t} e^C = C e^{-rk \, t}\]

\[|u+k| = C e^{-rk \, t}\]

Since \( k > 0 \) and \( u \) is small, \(|u+k| = u+k\) so

\[V + k = Ce^{-rk \, t}, \quad \text{using the initial condition,} \]

\[V(0) = 0 \] we get

\[0 + k = C e^{0} \quad \Leftrightarrow \quad k = C \cdot 0 \]

\[V = k e^{-rk \, t} - k = k (e^{-rk \, t} - 1)\]

To graph \( y = e^{-x} \) is like

\[y = 0 \quad \text{as } x \to \infty\]

So \( y = e^{-x} - 1 \) is like

\[y = 0 \quad \text{as } x \to \infty\]
so \( v = K(e^{-9Kt} - 1) \) would be like

\[
\begin{align*}
\text{Graph:} & \\
\text{Vertical axis:} & v \\
\text{Horizontal axis:} & t \\
\text{Point:} & (0, 0) \\
\text{Asymmetric:} & -K
\end{align*}
\]

3) Again \( \frac{dv}{dt} = r(v+k)(v-k) \Rightarrow -2rK(v+k) \) so

\[
\frac{dv}{v+k} = -2rKdt \quad \text{and again}
\]

\[
\ln|v+k| = -2rKt + C \quad \text{and again,} \quad C = K \text{ since } v(0) = 0
\]

So \( v+k = Ke^{-2rKt} \)

\[
v = -K + Ke^{-2rKt}
\]

\[
= -K(1 - e^{-2rKt})
\]

This isn't going to look very different from the previous

\[
\begin{align*}
\text{Graph:} & \\
\text{Vertical axis:} & v \\
\text{Horizontal axis:} & t \\
\text{Asymmetric:} & -K
\end{align*}
\]
\[ \lim_{t \to \infty} v(t) = \lim_{t \to \infty} -k(1 - e^{-kt}) \]
\[ = -k(1 - e^{-\infty}) = -k(1 - 0) = -k \checkmark \]

5) We get \[ \frac{\partial}{(v+k)(v-k)} = r \, dt \]

Substitute \[ \frac{1}{(v+k)(v-k)} = \frac{A}{v+k} + \frac{B}{v-k} = 0 \]

When \[ v = k \]
\[ 1 = A(v-k) + B(v+k) \]
\[ 50 \quad B = \frac{1}{2k} \]

When \[ v = -k \]
\[ 1 = A(-2k) + B(0) \]
\[ 50 \quad A = -\frac{1}{2k} \]

and get

\[ \int \frac{1}{2k} \left[ \frac{1}{v-k} - \frac{1}{v+k} \right] dv = \int r \, dt \quad \text{and we'll} \]

but put the \( \frac{1}{2k} \) on the right so

\[ \ln \left| \frac{v}{v-k} \right| - \ln |v+k| = 2krt + C \]

\[ \ln \left| \frac{v-k}{v+k} \right| = 2krt + C \quad \text{since } \ln a - \ln b = \ln \left( \frac{a}{b} \right) \]

and now for some heavy lifting with the algebra—

since I want \(-2krt\), rewrite this as

\[ -\ln \left| \frac{v-k}{v+k} \right| = -2krt + C \quad \text{and } \]
\[ \ln \left( \frac{1}{a} \right) = \ln (\frac{1}{a}) \quad \text{so this is} \]
\[ \ln \left| \frac{v+k}{v-k} \right| = -2\pi k t + C \]

Again, use the initial condition when \( t=0, v=0 \)

\[ \ln \left| \frac{0+k}{0-k} \right| = 0 + C \]

\[ 0 = 0 + C \quad c = 0 \quad \text{so} \]

\[ e^{\ln \left| \frac{v+k}{v-k} \right|} = e^{-2\pi k t} \]

\[ \left| \frac{v+k}{v-k} \right| = e^{-2\pi k t}, \quad \text{since} \quad -k < 0, \quad v < 0 \]

\( |v-k| = k-v \quad \text{so} \quad \frac{v+k}{k-v} = e^{-2\pi k t} \quad \text{call this} \quad e \]

\[ \frac{v+k}{k-v} = e \quad \text{so} \quad v+k = (k-v)e \]

\[ v+k = ke-v \]

\[ v+ve = ke-k \]

\[ v(1+e) = -k(1-e) \]

\[ v = -k \left[ \frac{1-e}{1+e} \right] \]

\[ v = -k \left[ \frac{1-e^{-2\pi k t}}{1+e^{-2\pi k t}} \right] \]

2) How you don't even need 'hospitals. Since \( e^{\infty} = 0 \)

And so

\[ \lim_{t \to \infty} v = -k \left[ \frac{1-0}{1+0} \right] = -k \]
8. There's a hard way to do this, and an easy way.

Hard way: take \( v = -1 \) \( \left[ \frac{1 - e^{-2\pi kt}}{1 + e^{-2\pi kt}} \right] \)

and differentiate twice. Ick.

Easy way: I already know \( \frac{dv}{dt} \):

\[
\frac{dv}{dt} = \sqrt{r v^2 - g}
\]

\[
\frac{d^2v}{dt^2} = \frac{1}{r} \left[ \frac{dv}{dt} \right] = \frac{1}{r} \left[ rv^2 - g \right]
\]

\[
= r \frac{1}{dt} (v^2) - \frac{1}{dt} (s)
\]

\[
= rv \frac{dv}{dt} - 0 = 2rv \left[ rv^2 - g \right]
\]

Chain rule:

\[
\frac{d^2v}{dt^2} = 2r^2v \left[ v^2 - \frac{g}{r} \right]
\]

Now:

\[
= 2r^2v \left[ v^2 - \frac{9}{r^2} \right]
\]

\[
= 2r^2v \left[ v^2 - k^2 \right]
\]

Now, let's check the sign of each term.

\( 2r^2 > 0 \) : +

\( v < 0 \) since \( v \) is falling -

\( v - k \) more negative since substrate \( k \) -

\( v + k \) since we are here in the picture.

\( v > -k \) so \( v + k > 0 \) +

\[ \text{So } \frac{d^2v}{dt^2} = + \left( - \right) \left( - \right) \left( + \right) = + \checkmark \]
9 a) \[ K = \sqrt{\frac{g}{r}} = \frac{\Delta}{S_{ud}} \]

\[ \frac{g}{r} = \frac{32 \, ft}{sec^2} \times \frac{5.18 \times 10^{-4}}{1} \times \frac{f + 1}{f + 1} = 6.178 \times 10^{-4} \times \frac{ft^2}{sec^2} \]

So \[ \sqrt{\frac{g}{r}} = \sqrt{6.178 \times 10^{-4} \, ft/sec} = 2.485 \times 10^{-2} \, ft/sec \]

\[ = 2.485 \times 10^{-2} \times (68.2) \frac{m}{hr} = -179.48 \, m/hr \]

So speed = -179 m/hr. Hitting ground that quickly would be fatal.

b) \[ \frac{g}{r} = \frac{32 \times 10^{-4} \, ft^2}{10.4 \, sec^2} \]

So \[ \sqrt{\frac{g}{r}} = \sqrt{3.88 \times 10^{-2} \, ft/sec} = 119.63 \, m/hr \]

Velocity about -120 m/hr, still fatal.

c) \[ \frac{g}{r} = \frac{32 \times 13 \times 10^{-2} \, ft^2}{sec^2} = 2.46 \times 10^2 \, ft^2/sec^2 \]

\[ \sqrt{\frac{g}{r}} = 15.68 \, ft/sec = 10.7 \, m/hr \]

This is something you can survive.