2.2 DOMAIN AND RANGE

In Example 4 on page 5, we defined \( R \) to be the average monthly rainfall at Chicago’s O’Hare airport in month \( t \). Although \( R \) is a function of \( t \), the value of \( R \) is not defined for every possible value of \( t \). For instance, it makes no sense to consider the value of \( R \) for \( t = -3 \), or \( t = 8.21 \), or \( t = 13 \) (since a year has 12 months). Thus, although \( R \) is a function of \( t \), this function is defined only for certain values of \( t \). Notice also that \( R \), the output value of this function, takes only the values \( \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\} \).

A function is often defined only for certain values of the independent variable. Also, the dependent variable often takes on only certain values. This leads to the following definitions:

If \( Q = f(t) \), then
- the **domain** of \( f \) is the set of input values, \( t \), which yield an output value.
- the **range** of \( f \) is the corresponding set of output values, \( Q \).

Thus, the domain of a function is the set of input values, and the range is the set of output values.

If the domain of a function is not specified, we usually assume that it is as large as possible—that is, all numbers that make sense as inputs for the function. For example, if there are no restrictions, the domain of the function \( f(x) = x^2 \) is the set of all real numbers, because we can substitute any real number into the formula \( f(x) = x^2 \). Sometimes, however, we may restrict the domain to suit a particular application. If the function \( f(x) = x^2 \) is used to represent the area of a square of side \( x \), we restrict the domain to positive numbers.

If a function is being used to model a real-world situation, the domain and range of the function are often determined by the constraints of the situation being modeled, as in the next example.

---

**Example 1**

The house painting function \( n = f(A) \) in Example 2 on page 4 has domain \( A > 0 \) because all houses have some positive area. There is a practical upper limit to \( A \) because houses cannot be infinitely large, but in principle, \( A \) can be as large or as small as we like, as long as it is positive. Therefore we take the domain of \( f \) to be \( A > 0 \).

The range of this function is \( n \geq 0 \), because we cannot use a negative amount of paint.

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**Choosing Realistic Domains and Ranges**

When a function is used to model a real situation, it may be necessary to modify the domain and range.

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**Example 2**

Algebraically speaking, the formula

\[
T = \frac{1}{4} R + 40
\]

can be used for all values of \( R \). If we know nothing more about this function than its formula, its domain is all real numbers. The formula for \( T = \frac{1}{4} R + 40 \) can return any value of \( T \) when we choose an appropriate \( R \)-value (See Figure 2.6.) Thus, the range of the function is also all real numbers. However, if we use this formula to represent the temperature, \( T \), as a function of a cricket’s chirp rate, \( R \), as we did in Example 1 on page 2, some values of \( R \) cannot be used. For example, it does not make sense to talk about a negative chirp rate. Also, there is some maximum chirp rate \( R_{\text{max}} \)
that no cricket can physically exceed. Thus, to use this formula to express $T$ as a function of $R$, we must restrict $R$ to the interval $0 \leq R \leq R_{\text{max}}$ shown in Figure 2.7.

![Figure 2.6: Graph showing that any $T$ value can be obtained from some $R$ value](image)

Figure 2.6: Graph showing that any $T$ value can be obtained from some $R$ value

![Figure 2.7: Graph showing that if $0 \leq R \leq R_{\text{max}}$, then $40 \leq T \leq T_{\text{max}}$](image)

Figure 2.7: Graph showing that if $0 \leq R \leq R_{\text{max}}$, then $40 \leq T \leq T_{\text{max}}$

The range of the cricket function is also restricted. Since the chirp rate is nonnegative, the smallest value of $T$ occurs when $R = 0$. This happens at $T = 40$. On the other hand, if the temperature gets too hot, the cricket will not be able to keep chirping faster. If the temperature $T_{\text{max}}$ corresponds to the chirp rate $R_{\text{max}}$, then the values of $T$ are restricted to the interval $40 \leq T \leq T_{\text{max}}$.

### Using a Graph to Find the Domain and Range of a Function

A good way to estimate the domain and range of a function is to examine its graph. The domain is the set of input values on the horizontal axis which give rise to a point on the graph; the range is the corresponding set of output values on the vertical axis.

#### Example 3

A sunflower plant is measured every day $t$, for $t \geq 0$. The height, $h(t)$ centimeters, of the plant\(^2\) can be modeled by using the logistic function

$$h(t) = \frac{260}{1 + 24(0.9)^t}.$$  

(a) Using a graphing calculator or computer, graph the height over 80 days.

(b) What is the domain of this function? What is the range? What does this tell you about the height of the sunflower?

#### Solution

(a) The logistic function is graphed in Figure 2.8.

![Figure 2.8: Height of sunflower as a function of time](image)

Figure 2.8: Height of sunflower as a function of time

(b) The domain of this function is \( t \geq 0 \). If we consider the fact that the sunflower dies at some point, then there is an upper bound on the domain, \( 0 \leq t \leq T \), where \( T \) is the day on which the sunflower dies.

To find the range, notice that the smallest value of \( h \) occurs at \( t = 0 \). Evaluating gives \( h(0) = 10.4 \) cm. This means that the plant was 10.4 cm high when it was first measured on day \( t = 0 \). Tracing along the graph, \( h(t) \) increases. As \( t \)-values get large, \( h(t) \)-values approach, but never reach, 260. This suggests that the range is \( 10.4 \leq h(t) < 260 \). This information tells us that sunflowers typically grow to a height of about 260 cm.

**Using a Formula to Find the Domain and Range of a Function**

When a function is defined by a formula, its domain and range can often be determined by examining the formula algebraically.

**Example 4**  
State the domain and range of \( g \), where

\[
g(x) = \frac{1}{x}.
\]

**Solution**  
The domain is all real numbers except those which do not yield an output value. The expression \( 1/x \) is defined for any real number \( x \) except 0 (division by 0 is undefined). Therefore,

Domain: all real \( x \), \( x \neq 0 \).

The range is all real numbers that the formula can return as output values. It is not possible for \( g(x) \) to equal zero, since 1 divided by a real number is never zero. All real numbers except 0 are possible output values, since all nonzero real numbers have reciprocals. Thus

Range: all real values, \( g(x) \neq 0 \).

The graph in Figure 2.9 indicates agreement with these values for the domain and range.

![Graph of \( g(x) = \frac{1}{x} \)](image)

**Figure 2.9:** Domain and range of \( g(x) = 1/x \)

**Example 5**  
Find the domain of the function \( f(x) = \frac{1}{\sqrt{x-4}} \) by examining its formula.

**Solution**  
The domain is all real numbers except those for which the function is undefined. The square root of a negative number is undefined (if we restrict ourselves to real numbers), and so is division by zero. Therefore we need

\[
x - 4 > 0.
\]

Thus, the domain is all real numbers greater than 4.

Domain: \( x > 4 \).

In Example 6 on page 63, we saw that for \( f(x) = 1/\sqrt{x-4} \), the output, \( f(x) \), cannot be negative. Note that \( f(x) \) cannot be zero either. (Why?) The range of \( f(x) = 1/\sqrt{x-4} \) is \( f(x) > 0 \). See Problem 16.
Exercises and Problems for Section 2.2

Exercises

In Exercises 1–4, use a graph to find the range of the function on the given domain.

1. \( f(x) = \frac{1}{x} \), \(-2 \leq x \leq 2\)
2. \( f(x) = \frac{1}{x^2} \), \(-1 \leq x \leq 1\)
3. \( f(x) = x^2 - 4 \), \(-2 \leq x \leq 3\)
4. \( f(x) = \sqrt{9 - x^2} \), \(-3 \leq x \leq 1\)

Graph and give the domain and range of the functions in Exercises 5–12.

5. \( f(x) = (x - 4)^3 \)
6. \( f(x) = x^2 - 4 \)
7. \( f(x) = 9 - x^2 \)
8. \( f(x) = x^3 + 2 \)

Problems

Find the domain and range of functions in Exercises 15–18 algebraically.

9. \( f(x) = \sqrt{8 - x} \)
10. \( f(x) = \sqrt{x - 3} \)
11. \( f(x) = \frac{-1}{(x + 1)^2} \)
12. \( f(x) = \frac{1}{x^2} \)

In Exercises 19–24, find the domain and range.

13. \( q(x) = \sqrt{x^2 - 9} \)
14. \( f(x) = \frac{1}{\sqrt{x - 4}} \)
15. \( m(x) = 9 - x \)
16. \( n(x) = 9 - x^4 \)

17. \( f(x) = -x^2 + 7 \)
18. \( f(x) = \sqrt{x + 7} \)
19. \( f(x) = x^2 + 2 \)
20. \( f(x) = \frac{1}{(x + 1)^2} + 3 \)
21. \( f(x) = x - 3 \)
22. \( f(x) = (x - 3)^2 + 2 \)

23. Give a formula for a function whose domain is all non-negative values of \( x \) except \( x = 3 \).

24. Give a formula for a function that is undefined for \( x = 8 \) and for \( x < 4 \), but is defined everywhere else.

25. A restaurant is open from 2 pm to 2 am each day, and a maximum of 200 clients can fit inside. If \( f(t) \) is the number of clients in the restaurant \( r \) hours after 2 pm each day, what are a reasonable domain and range for \( f(t) \)?

26. What is the domain of the function \( f \) giving average monthly rainfall at Chicago’s O’Hare airport? (See Table 1.2 on page 5)

27. A movie theater seats 200 people. For any particular show, the amount of money the theater makes is a function of the number of people, \( n \), in attendance. If a ticket costs \$4.00, find the domain and range of this function. Sketch its graph.

28. A car gets the best mileage at intermediate speeds. Graph the gas mileage as a function of speed. Determine a reasonable domain and range for the function and justify your reasoning.

30. A car gets the best mileage at intermediate speeds. Graph the gas mileage as a function of speed. Determine a reasonable domain and range for the function and justify your reasoning.

31. (a) Use Table 2.7 to determine the number of calories that a person weighing 200 lb uses in a half-hour of walking.\(^3\)

(b) Table 2.7 illustrates a relationship between the number of calories used per minute walking and a person’s weight in pounds. Describe in words what is true about this relationship. Identify the dependent and independent variables. Specify whether it is an increasing or decreasing function.

(c) (i) Graph the linear function for walking, as described in part (b), and estimate its equation.

(ii) Interpret the meaning of the vertical intercept of the graph of the function.

(iii) Specify a meaningful domain and range for your function.

(iv) Use your function to determine how many calories per minute a person who weighs 135 lb uses per minute of walking.

Table 2.7 Calories per minute as a function of weight

<table>
<thead>
<tr>
<th>Activity</th>
<th>100 lb</th>
<th>120 lb</th>
<th>150 lb</th>
<th>170 lb</th>
<th>200 lb</th>
<th>220 lb</th>
</tr>
</thead>
<tbody>
<tr>
<td>Walking</td>
<td>2.7</td>
<td>3.2</td>
<td>4.0</td>
<td>4.6</td>
<td>5.4</td>
<td>5.9</td>
</tr>
<tr>
<td>Bicycling</td>
<td>5.4</td>
<td>6.5</td>
<td>8.1</td>
<td>9.2</td>
<td>10.8</td>
<td>11.9</td>
</tr>
<tr>
<td>Swimming</td>
<td>5.8</td>
<td>6.9</td>
<td>8.7</td>
<td>9.8</td>
<td>11.6</td>
<td>12.7</td>
</tr>
</tbody>
</table>

\(^3\)Source: 1993 World Almanac. Speeds assumed are 3 mph for walking, 10 mph for bicycling, and 2 mph for swimming.
In Exercises 32–33, find the domain and range.

32. \( g(x) = a + 1/x \), where \( a \) is a constant

33. \( q(r) = (x - b)^{1/2} + 6 \), where \( b \) is a constant

34. The last digit, \( d \), of a phone number is a function of \( n \), its position in the phone book. Table 2.8 gives \( d \) for the first 10 listings in the 1998 Boston telephone directory. The table shows that the last digit of the first listing is 3, the last digit of the second listing is 8, and so on. In principle we could use a phone book to figure out other values of \( d \). For instance, if \( n = 300 \), we could count down to the 300th listing in order to determine \( d \). So we write \( d = f(n) \).

(a) What is the value of \( f(6) \)?

(b) Explain how you could use the phone book to find the domain of \( f \).

(c) What is the range of \( f \)?

Table 2.8

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( d )</td>
<td>3</td>
<td>8</td>
<td>4</td>
<td>0</td>
<td>1</td>
<td>8</td>
<td>0</td>
<td>4</td>
<td>3</td>
<td>5</td>
</tr>
</tbody>
</table>

35. In month \( t = 0 \), a small group of rabbits escapes from a ship onto an island where there are no rabbits. The island rabbit population, \( p(t) \), in month \( t \) is given by

\[
p(t) = \frac{1000}{1 + 19(0.9)^t}, \quad t \geq 0.
\]

(a) Evaluate \( p(0) \), \( p(10) \), \( p(50) \), and explain their meaning in terms of rabbits.

(b) Graph \( p(t) \) for \( 0 \leq t \leq 100 \). Describe the graph in words. Does it suggest the growth in population you would expect among rabbits on an island?

(c) Estimate the range of \( p(t) \). What does this tell you about the rabbit population?

(d) Explain how you can find the range of \( p(t) \) from its formula.

36. Bronze is an alloy or mixture of the metals copper and tin. The properties of bronze depend on the percentage of copper in the mix. A chemist decides to study the properties of a given alloy of bronze as the proportion of copper is varied. She starts with 9 kg of bronze that contain 3 kg of copper and 6 kg of tin and either adds or removes copper. Let \( f(x) \) be the percentage of copper in the mix if \( x \) kg of copper are added \( (x > 0) \) or removed \( (x < 0) \).

(a) State the domain and range of \( f \). What does your answer mean in the context of bronze?

(b) Find a formula in terms of \( x \) for \( f(x) \).

(c) If the formula you found in part (b) was not intended to represent the percentage of copper in an alloy of bronze, but instead simply defined an abstract mathematical function, what would be the domain and range of this function?

2.3 PIECEWISE DEFINED FUNCTIONS

A function may employ different formulas on different parts of its domain. Such a function is said to be piecewise defined. For example, the function graphed in Figure 2.10 has the following formulas:

\[
y = x^2 \quad \text{for } x \leq 2 \quad \text{or more compactly} \quad y = \begin{cases} 
  x^2 & \text{for } x \leq 2 \\
  6 - x & \text{for } x > 2 
\end{cases}
\]

![Figure 2.10: Piecewise defined function](image)

**Example 1**

Graph the function \( y = g(x) \) given by the following formulas:

\[
g(x) = x + 1 \quad \text{for } x \leq 2 \quad \text{and} \quad g(x) = 1 \quad \text{for } x > 2.
\]

Using bracket notation, this function is written:

\[
g(x) = \begin{cases} 
  x + 1 & \text{for } x \leq 2 \\
  1 & \text{for } x > 2
\end{cases}
\]