Homework Quiz 7

Tuesday, July 31th

Name: Solution 5

Answer all three questions. You may use your HW on this quiz. You may not use a calculator.

1) (4 points) [Ch 5, #39] If X is an exponential random variable with parameter \( \lambda = 1 \), compute the probability density function of the random variable \( Y \) defined by \( Y = \log(X) \). (Here, \( \log(X) = \ln(X) \)).

\[
F_X(a) = \begin{cases} e^{-a} & a \geq 0 \\ 0 & \text{otherwise} \end{cases}
\]

\[
F_Y(a) = P\{Y \leq a\} = P\{\log(x) \leq a\} = P\{x \leq e^a\} = F_X(e^a)
\]

\[
F_Y(a) = F_X(e^a)
\]

\[
f_Y(a) = e^a f_X(e^a)
\]

Since \( e^a > 0 \) for all \( a \),

\[
e^a f_X(e^a) = e^a e^{-e^a} = f_Y(a)
\]

for all \( a \).
2) (4 points) [Ch 5, #32] The time (in hours) required to repair a machine is an exponentially distributed random variable with parameter \( \lambda = 1/2 \). What is:

a) the probability that a repair time exceeds 2 hours?

\[
    f_X(x) = \begin{cases} 
        \frac{1}{2}e^{-\frac{x}{2}} & \text{if } x \geq 0 \\
        0 & \text{otherwise} 
    \end{cases}
\]

\[
    \Pr(X > 2) = 1 - \Pr(X < 2) = 1 - \int_0^2 \frac{1}{2}e^{-\frac{x}{2}} \, dx = 1 - \left[-e^{-\frac{x}{2}} + e^0\right] = e^{-1}
\]

b) the conditionally probability that a repair takes at least 10 hours, given that its duration exceeds 9 hours?

Since time is exponentially distributed R.V., it is memoryless. This means:

\[
    \Pr(X > 10 | P > 9) = \Pr(X > 13) = 1 - \Pr(X < 13) = 1 - \int_0^{13} \frac{1}{2}e^{-\frac{x}{2}} \, dx
\]

\[
    = 1 - \left[-e^{-\frac{x}{2}} + e^0\right] = e^{-1/2}
\]

3) (2 points) [Ch 5, #38] If \( Y \) is uniformly distributed over (0, 5), what is the probability that the roots of \( 4x^2 + 4xY + Y + 2 = 0 \) are both real?

The roots are found using the Quadratic Formula,

\[
    x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

The roots are real \( \iff b^2 - 4ac > 0 \)

\[
    \iff (4Y)^2 - 4(4)(Y+2) > 0
\]

\[
    16Y^2 - 16Y - 32 > 0
\]

\[
    16(Y^2 - Y - 2) > 0
\]

\[
    16(Y-2)(Y+1) > 0
\]

\[
    \Pr(16(Y-2)(Y+1) > 0) = \Pr(Y > 2) = \frac{3}{5}
\]