Question 1. By constructing a multiplication table, or otherwise, determine which of the following statements is correct:

(i) the set \{1, 9, 13, 17\} is a group under multiplication mod 20:

(ii) the set \{1, 7, 13\} is a group under multiplication mod 14.

Question 2. Fix a positive integer \(n\). Recall that \(\phi(n)\) denotes the number of integers \(k\), \(0 < k < n\), such that \(k\) is relatively prime to \(n\). A famous theorem of Euler states that if \(a\) is an integer relatively prime to \(n\), then \(a^{\phi(n)} \equiv 1(\text{mod } n)\). Use the following steps to give a simple proof of this result.

(i) Prove that if \(G\) is a finite group with identity \(e\) and \(g\) is an arbitrary element of \(G\), then \(g^{|G|} = e\) where \(|G|\) denotes the order of \(G\).

(ii) Now let \(G\) be the set of all integers \(m\), \(0 < m < n\), such that \(m\) is relatively prime to \(n\). Show that \(G\) is a group under multiplication mod \(n\).

(iii) Deduce from (ii) that \(a^{\phi(n)} \equiv 1(\text{mod } n)\) for any \(a\) in \(G\).

(iv) Use (iii) to show that if \(a\) is any integer relatively prime to \(n\), then \(a^{\phi(n)} \equiv 1(\text{mod } n)\) (Remember that (iii) only applies to integers \(a\) such that \(0 < a < n\).)

Question 3. Let \(H\) be a subgroup of a group \(G\) and let \(\equiv_H\) denote the equivalence relation on \(G\) defined by

\[ a \equiv_H b \iff b^{-1}a \in H. \]

Show that for an element \(a\) in \(G\) the equivalence class

\[ [a] = \{b \in G : a \equiv_H b\} \]

containing \(a\) has the property that

\[ [a] = aH = \{ah : h \in H\}. \]
Now let $G$ be the group of all symmetries of an equilateral triangle in figure 1.

**Question 4.** Describe the group $G$ in words. What is the order of $G$? Then describe it algebraically by giving letters to each of the elements of $G$ and write down the multiplication table of $G$. Use $G$ to label the 6 congruent triangles in figure 1.

**Question 5.** Describe in words and algebraically the symmetry group $H$ of the graphic design in figure 2. What is its order?

**Question 6.** Prove that there are 2 left cosets of $H$ in $G$. Describe these two cosets.

**Question 7.** Determine if the subgroup $H$ has the property that $gHg^{-1} = H$ for all $g$ in $G$. 