As we began to look at in class, one of the basic results in group theory is the result appearing as theorem 2.7.6 in the text. We'll prove the result in class, but it is instructive to look at what the theorem means when applied to symmetry groups of various tilings.

Recall first some definitions. A subgroup $H$ of a group $G$ is said to be a normal subgroup when $gHg^{-1} \subseteq H$ for all $g$ in $G$. Now let $\pi : G \rightarrow \tilde{G}$ be a homomorphism from one group $G$ to a possibly different group $\tilde{G}$; for the moment, neither $G$ nor $\tilde{G}$ need be a finite group and $\pi$ need not be surjective.

**Question 1.** Let $\pi : G \rightarrow \tilde{G}$ be a homomorphism from one group $G$ to a possibly different group $\tilde{G}$. Show that

(i) if $e$ is the identity in $G$ and $\tilde{e}$ is the identity in $\tilde{G}$, then $\pi(e) = \tilde{e}$,

(ii) $\pi(g^{-1}) = \pi(g)^{-1}$ for each $g$ in $G$.

Now recall that the kernel, $\ker(\pi)$, of $\pi$ is the subset of $G$ defined by

$$\ker(\pi) = \left\{ g \in G : \pi(g) = \tilde{e} \right\}.$$

**Question 2.** Show that $\ker(\pi)$ is always a normal subgroup of $G$.

As an illustration, consider the homomorphism $\pi : \mathbb{Z} \rightarrow \mathbb{Z}_N$ defined by $\pi : m \rightarrow [m]$ where $[m]$ is the equivalence class

$$[m] = \left\{ k \in \mathbb{Z} : N \mid (m - k) \right\}.$$

This admits a natural generalization.

**Question 3.** Fix integers $n, N$ and let $\mathbb{Z}_n, \mathbb{Z}_N$ be the usual groups of integers in which the group operation is addition modulo $n$ and $N$ respectively.

(i) Show that $m \rightarrow [m]$ defines a homomorphism $\pi_{n,N}$ from $\mathbb{Z}_n$ onto $\mathbb{Z}_N$ whenever $n$ is a multiple of $N$,

(ii) determine the kernel of this homomorphism $\pi_{n,N}$.

Now let's look at the family of dihedral groups $D_n$ again. By convention we'll denote by $D_1$ the group $\{e, r\}$ generated by a single element $r$ such that $r^2 = e$; we can think of $D_1$
as the symmetry group of the interval $[-1, 1]$ centered at the origin in the number line. Notice that
\[ e \to 0, \quad r \to 1 \]
is an isomorphism from $D_1$ onto the additive group $\mathbb{Z}_2$ of integers mod 2. A formal algebraic definition of $D_n$ is convenient (a strategy adopted by the text on page 106). So, we’ll denote by $D_n$ the group generated by two elements $\{a, r\}$ such that
\[ a^2 = e, \quad r^n = e, \quad r^k a = a r^{n-k}. \]

Use this definition of $D_n$ in answering the following question.

**Question 4.** For the dihedral group $D_n$ show that

(i) the group
\[ D_n = \{ e, r, r^2, \ldots, r^{n-1}, a, ar, ar^2, \ldots, ar^{n-1} \}, \]
and hence that $|D_n| = 2n$,

(ii) the cyclic subgroup
\[ C_n = \{ e, r, r^2, \ldots, r^{n-1} \} \]
generated by $r$ is a normal subgroup of $D_n$,

(iii) the subgroup $A_n = \{ e, a \}$ generated by $a$ is not normal in $D_n$,

(iv) when $R$ denotes rotation through $\frac{2\pi}{n}$ about the origin in the plane and $A$ denotes rotation about the $x$-axis in 3-space, then the mapping $r \to R$ and $a \to A$ defines an isomorphism from $D_n$ onto the symmetry group of a regular $n$-gon centered at the origin in the $x$-$y$ plane in 3-space for each $n \geq 3$.

What’s missing from this discussion is a symmetry group identification of $D_2$. For this consider a rectangle

\[ \text{figure 1.} \]

which we’ll think of as being centered at the origin in the $x$-$y$ plane in 3-space.
Question 5. Denote rotation about the $x$-axis in 3-space by $A$ and rotation about the $y$-axis by $B$. Show that

(i) the symmetry group, $G_\square$, of the rectangle in figure 1 is generated by $\{A, B\}$ and write down its multiplication table,

(ii) the mapping $a \rightarrow A$, $r \rightarrow AB$ defines an isomorphism from $D_2$ onto $G_\square$.

One advantage of realizing the dihedral groups algebraically is that we can then define homomorphisms very easily. You’ve already met a similar one in earlier homework.

Question 6. Fix integers $n$, $N$ such that $n$ is a multiple of $N$ and denote by $k \rightarrow [k]$ the homomorphism $\pi_{nN} : n \rightarrow [n]$ from $\mathbb{Z}_n$ onto $\mathbb{Z}_N$ introduced in Question 3. Show that

$$r^k \rightarrow r^{[k]}, \; ar^m \rightarrow ar^{[m]}$$

defines a homomorphism from $D_n$ onto $D_N$.

Now let’s try to illustrate such mappings in terms of symmetry groups of various designs. Consider the following regular octagon

![Regular Octagon](image)

which has been tiled by 16 congruent right triangles labelled from 0 to 15. Now let’s color these triangles systematically resulting in the ‘Roman Candle’
Question 7. (i) What is the symmetry group of the Roman Candle in figure 3?

(ii) When the triangles in figure 1 are labelled 0, 1, 2, \ldots, 15 as shown, show that \( m \rightarrow [m] \) defines a homomorphism from the symmetry group \( D_8 \) of the regular octagon in figure 1 onto \( D_1 \).

(iii) What normal subgroup of \( D_8 \) is the kernel of this homomorphism?

(iv) What, if any, is the relation of the Roman candle to this homomorphism?

Okay, now it’s your turn to be a Euclidean Escher. Below is a rectangle tiled with 4 congruent rectangles

\[ \begin{array}{cc}
\text{figure 4.}
\end{array} \]
Below is another regular octagon tiled with 16 congruent right triangles.

\[ \text{figure 5.} \]

**Question 8.** (i) Write down algebraically a homomorphism \( \pi : D_8 \rightarrow D_2 \) and determine its kernel.

(ii) Color the rectangles and the right triangles in figures 4 and 5 to illustrate your homomorphism and kernel in part (i) of this question.