This practice sheet has more problems than the actual exam which will only have about 7-8 problems. The main concepts the exam covers are: definition of a group, subgroup, order of an element, order of a subgroup, index of a subgroup, cyclic subgroup, abelian group, symmetric groups, cosets, normal subgroup, quotient group, homomorphism, automorphism, isomorphism. It can be helpful to put away the book, recall from memory the definitions of these concepts and the main results (such as Lagrange’s Theorem that the order of a subgroup divides the order of the group).

One more thing - please bring blank paper to the exam for writing your answers on.

1. Let $G$ be the group $G = \{e, a, b, ab\}$ such that $a^2 = b^2 = e, ab = ba$. Prove that $\text{Aut}(G)$ is isomorphic to $S_3$, the symmetric group of 3 things.

2. Suppose $x, y, z \in G$ and $xyz = 1$. Does it follow that $yzx = 1$? If not, find an explicit counterexample.

3. Does $\mathbb{R}/\mathbb{Z}$ have a subgroup of order 10? If so, find one. If not, explain.

4. Let $n, m$ be positive integers. Show that there exists a surjective homomorphism $\phi: \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ if and only if $m \mid n$ (this means $n/m$ is an integer).

5. Show that if $H < G$ is a subgroup and $g \in G$ then $gHg^{-1}$ is a subgroup of $G$ and $\text{ind}_G(H) = \text{ind}_G(gHg^{-1})$.

6. Let $Z(G) = \{g \in G : gh = hg \text{ for every } h \in G\}$ be the center of $G$. Suppose that $G/Z(G)$ is cyclic. Show that $G$ must be abelian.

7. Let $g \in G$ be an element of order 1000. What is the order of $g^{15}$? Prove your answer.

8. Let $a, b \in G$. Prove that $(aba^{-1})^n = ab^n a^{-1}$ for every integer $n$.

9. Let $\phi: G \to H$ be a homomorphism. Suppose $N$ is a normal subgroup of $H$. Prove that $\phi^{-1}(N) = \{g \in G : \phi(g) \in N\}$ is a normal subgroup of $G$.

10. Recall that $S_{n+1}$ is the group of all permutations of the set $\{1, \ldots, n+1\}$. Let $S_n$ be the subgroup of $S_{n+1}$ of all permutations that fix $n+1$. In other words, $S_n = \{g \in S_{n+1} : (n+1)g = n+1\}$. Is $S_n$ normal in $S_{n+1}$? Prove your answer.

11. Show $\text{Aut}(S_3)$ is isomorphic with $S_3$.

12. Suppose $H < G$ is a cyclic subgroup which is normal in $G$. Show that every subgroup of $H$ is normal in $G$.

13. Let $H, N$ be subgroups of $G$ and let $HN = \{hn : h \in H, n \in N\}$. Suppose that $N$ is normal in $G$. Show that $HN$ is a subgroup of $G$. 