Example: Let $A$ be the region in the plane given by

$$\{(x,y) \mid 4x^2 + y^2 \leq 4\}$$

Find the absolute extremes for $f(x,y) = x^2 - x + y^2$ over $A$.

$A$ is the region bounded by ellipse $x^2 + \frac{y^2}{4} = 1$ (graph equality, test each side for inequality).

(1) Critical pts in interior of $A$

$$0 = \frac{\partial f}{\partial x} = 2x - 1 \Rightarrow x = \frac{1}{2}$$

$$0 = \frac{\partial f}{\partial y} = 2y \Rightarrow y = 0$$

$$f\left(\frac{1}{2},0\right) = -\frac{1}{4}$$

(1/2,0) only critical pt

(2) Extremes along boundary

We write the boundary of as $l_1 \cup l_2$, where $l_1$ is upper half of ellipse, $l_2$ is lower half.

Extremes on $l_1$: $y^2 = 4 - 4x^2 \Rightarrow y = \sqrt{4 - 4x^2}, -1 \leq x \leq 1$

$$l_1: (x, \sqrt{4 - 4x^2}) \quad -1 \leq x \leq 1$$
\[ f(x, \sqrt{4-4x^2}) = x^2 - x + 4 - 4x^2 = 4 - x - 3x^2 \]

\[ f'(x) = -1 - 6x \implies x = -\frac{1}{6} \]

Check: \( x = -1, -\frac{1}{6}, 1 \)

\[ f(-1, 0) = 2, \quad f(1, 0) = 0 \]

\[ f\left(-\frac{1}{6}, \sqrt{\frac{21}{36}}\right) = 4 + \frac{1}{6} - \frac{3}{36} = 4\frac{1}{12} \]

\[ y = -\sqrt{4-4x^2} \quad -1 \leq x \leq 1 \]

\[ (x, -\sqrt{4-4x^2}) \quad -1 \leq x \leq 1 \]

\[ f(x, -\sqrt{4-4x^2}) = x^2 - x + 4 - 4x^2 \quad \text{same as above} \]

Check: \( x = -1, -\frac{1}{6}, 1 \)

\[ f(-1, 0) = 2, \quad f(1, 0) = 0, \quad f\left(-\frac{1}{6}, -\sqrt{\frac{21}{36}}\right) = 4\frac{1}{12} \]

Absolute max = 4\frac{1}{12} occurring at \((-\frac{1}{6}, \frac{\sqrt{21}}{36})\)

\((-\frac{1}{6}, -\frac{\sqrt{21}}{36})\)

Absolute min = -\frac{1}{4} occurring at \((\frac{1}{2}, 0)\)
Example  Let $A = \{(x, y) \mid 4x^2 + y^2 \leq 4 \text{ and } x \geq 0\}$.

Find the absolute extrema of 

$f(x, y) = x^2 - x + y^2$ on $A$.

Same region as before.  
Half the region of before.

(1) Critical pts in int. of $A$

Same as before: $(\frac{1}{2}, 0)$  
$f(\frac{1}{2}, 0) = -\frac{1}{4}$

2) Extrema on boundary of $A$:

1. $l_1 : -2 \leq y \leq 2$  
   $x = \sqrt{1 - y^2}$

2. $l_2 : -2 \leq y \leq 2$  
   $x = 0$

$l_1 : f(\sqrt{1 - y^2}, y) = 1 - \frac{y^2}{4} - \sqrt{1 - \frac{y^2}{4}} + y^2$

$= 1 + \frac{3}{4} y^2 - \frac{1}{2} \sqrt{1 - \frac{y^2}{4}}$

$f' = \frac{3}{2} y - \frac{1}{2} (1 - \frac{y^2}{4})^{-\frac{1}{2}} (-\frac{y}{2})$
Critical pts: \[ 0 = \frac{y}{7} \left( 6 + \frac{1}{1 - e^{-y^2}} \right) \Rightarrow y = 0 \]

Also \( y = -2, 2 \) are critical pt because \( f'(y) \)undefined

And: \( f(-2, 0) = 4 \quad f(0, 2) = 4 \quad f(0, 0) = 0 \)

Absolute max: \( 4 = f(0, -2) = f(0, 2) \)

Absolute min: \( 0 = f(0, 0) \)