Chapter 20
Brownian Motion and Itô’s Lemma

Question 20.1.
If \( y = \ln(S) \) then \( S = e^y \) and \( dy = \left( \frac{\alpha(S,t)}{S} - \frac{\sigma(S,t)^2}{2S^2} \right) dt + \frac{\sigma(S,t)}{S} dZ_t \),

a) \( dy = \left( \frac{\alpha}{e^y} - \frac{\sigma^2}{2e^{2y}} \right) dt + \frac{\sigma}{e^y} dZ_t. \)

b) \( dy = \left( \frac{\lambda a}{e^y} - \lambda - \frac{\sigma^2}{2e^{2y}} \right) dt + \frac{\sigma}{e^y} dZ_t. \)

c) \( dy = \left( \alpha - \frac{\sigma^2}{2} \right) dt + \sigma dZ_t. \)

Question 20.2.
If \( y = S^2 \) then \( S = \sqrt{y} \) and \( dy = \left( 2S \alpha (S,t) + \sigma (S,t)^2 \right) dt + 2S \sigma (S,t) dZ_t \) where \( \alpha (S,t) \) is the drift of \( S \) and \( \sigma (S,t) \) is the volatility of \( S \). For the three specifications:

a) \( dy = \left( 2\alpha \sqrt{y} + \sigma^2 \right) dt + 2\sqrt{y} \sigma dZ_t. \)

b) \( dy = \left( 2\sqrt{y} \lambda (a - \sqrt{y}) + \sigma^2 \right) dt + 2\sqrt{y} \sigma dZ_t \) \( \tag{1} \)
\( = \left( 2\lambda a \sqrt{y} - 2\lambda y + \sigma^2 \right) dt + 2\sqrt{y} \sigma dZ_t. \) \( \tag{2} \)

c) \( dy = \left( 2\alpha + \sigma^2 \right) ydt + 2\sigma ydZ_t. \)

Question 20.3.
If \( y = 1/S \) then \( S = 1/y \) and \( dy = \left( -S^{-2} \alpha (S,t) + S^{-3} \sigma (S,t)^2 \right) dt - S^{-2} \sigma (S,t) dZ_t, \)

a) \( dy = \left( -\alpha y^2 + \sigma^2 y^3 \right) dt - \sigma y^2 dZ_t. \)

b) \( dy = \left( -\lambda (ay^2 - y) + \sigma^2 y^3 \right) dt - \sigma y^2 dZ_t. \)

c) \( dy = \left( -\alpha + \sigma^2 \right) ydt - \sigma ydZ_t. \)
Question 20.4.

If \( y = \sqrt{S} \) then \( S = y^2 \) and

\[
dy = \left( \frac{1}{2} S^{-1/2} \alpha (S, t) - \frac{1}{8} S^{-3/2} \sigma (S, t)^2 \right) dt + \frac{1}{2} S^{-1/2} \sigma (S, t) dZ_t
\]

\[
= \left( \frac{1}{2y} \alpha (S, t) - \frac{1}{8y^3} \sigma (S, t)^2 \right) dt + \frac{1}{2y} \sigma (S, t) dZ_t
\]

a) \( dy = \left( \frac{\alpha}{2y} - \frac{\sigma^2}{8y^3} \right) dt + \frac{\sigma}{2y} dZ_t. \)

b) \( dy = \left( \frac{\lambda \alpha}{2y} - \frac{\lambda}{2y^2} - \frac{\sigma^2}{8y^3} \right) dt + \frac{\sigma}{2y} dZ_t. \)

c) \( dy = \left( \frac{\alpha}{2} - \frac{\sigma^2}{8} \right) y dt + \frac{\sigma}{2} y dZ_t. \)

Question 20.5.

Let \( y = S^2 Q^{0.5} \), then

\[
\frac{dy}{y} = \left( 2 (\alpha S - \delta S) + \frac{\alpha Q - \delta Q}{2} + \sigma^2_S - \frac{\sigma^2_Q}{8} + \rho \sigma_S \sigma_Q \right) dt + 2 \sigma_S dZ_S + b \sigma_Q dZ_Q.
\]

Question 20.6.

If \( y = \ln (S Q) = \ln (S) + \ln (Q) \) then

\[
dy = d \ln (S) + d \ln (Q)
\]

\[
= \left( \alpha S - \delta S - \frac{\sigma^2_S}{2} + \alpha Q - \delta Q - \frac{\sigma^2_Q}{2} \right) dt + \sigma_S dZ_S + \sigma_Q dZ_Q.
\]

Question 20.7.

With \( \delta = 0 \), the prepaid forward price for \( S^a \) is

\[
F_{0,1}^P (S^a) = S_0^a \exp \left( (a - 1) r + \frac{1}{2} a (a - 1) \sigma^2 \right).
\]

a) If \( a = 2 \), \( F_{0,1}^P (S^2) = 100^2 \exp (.06 + .4^2) = 12461. \)

b) If \( a = .5 \), \( F_{0,1}^P (S^{.5}) = 10 \exp \left( -.03 - \frac{.4^2}{8} \right) = 9.5123. \)

c) If \( a = -2 \), \( F_{0,1}^P (S^{-2}) = 100^{-2} \exp \left( -.18 + 3 (.4^2) \right) = 1.3499 \times 10^{-4}. \)
**Question 20.8.**

Since the process \( y = S^a Q^b \) follows geometric Brownian motion, i.e. \( dy = \alpha_y y dt + \sigma_y y dZ_y \) the price of the claims will be \( e^{-r} E^* (y_1) = y_0 e^{(\alpha_y - r)} \). We use Itô’s lemma, as in equation (20.38), with \( \delta = 0 \) and \( \alpha_S = \alpha_Q = r \) to arrive at the drift

\[
\alpha_y = ar + br + \frac{1}{2} a (a - 1) \sigma_S^2 + \frac{1}{2} b (b - 1) \sigma_Q^2 + ab \rho \sigma_S \sigma_Q
\]

(9)

\[
= .06 (a + b) + \frac{4^2}{2} a (a - 1) + \frac{.2^2}{2} b (b - 1) - .3 (.4)(.2) ab.
\]

(10)

a) Since \( a = b = 1, \ y_0 = 10000 \) and \( \alpha_y = .12 - .024 = .096 \) hence the claim is worth \( 10000e^{-0.096} = 10366.56 \).

b) Since \( a = 1 \) and \( b = -1, \ y_0 = 1 \) and \( \alpha_y = .2^2 + .024 = .064 \) hence the claim is worth \( e^{.064} = 1.004 \).

c) Since \( a = 1/2 \) and \( b = 1/2, y_0 = 100 \) and \( \alpha_y = .029 \) hence the claim is worth \( 100e^{.029} = 96.948 \).

d) Since \( a = -1 \) and \( b = -1, \ y_0 = 1/10000 \) and \( \alpha_y = .056 \) hence the claim is worth \( (e^{.056})/10000 = 9.9601 \times 10^{-5} \).

e) Since \( a = 2 \) and \( b = 1, \ y_0 = 1000000 \) and \( \alpha_y = .292 \) hence the claim is worth \( 1000000e^{.292} = 1.2612 \) million.

**Question 20.9.**

It is obvious if \( t = 0 \) the proposed solution will be equal to \( X_0 \). It is helpful to rewrite the solution as

\[
X_t = e^{-\lambda t} \left( X_0 + a (e^{\lambda t} - 1) + \sigma \int_0^t e^{\lambda s} dZ_s \right) = e^{-\lambda t} Y_t
\]

(11)

where \( dY_t = a \lambda e^{\lambda t} dt + \sigma e^{\lambda t} dZ_t \). Since \( e^{-\lambda t} \) is deterministic,

\[
dX_t = (-\lambda e^{-\lambda t} dt) Y_t + e^{-\lambda t} dY_t = (a \lambda - \lambda X_t) dt + \sigma dZ_t.
\]

(12)

**Question 20.10.**

Note that if \( V(S) \) satisfies the given equation, then

\[
E^* (dV) = \left[ (r - \delta) SVS + \frac{1}{2} \sigma^2 S^2 V_{SS} \right] dt = r V dt.
\]

(13)
Since $V(S) = k S^{h_1}$ where $a$ is constant, showing $y = S^a$ satisfies $E^*(dy) = rydt$ when $a = h_1$ is sufficient (i.e. the constant term is irrelevant). Using Ito’s lemma,

$$E^*(dy) = aS^{a-1}(r - \delta) S + \frac{1}{2}a (a - 1) S^{a-2} \sigma^2 S^2$$

$$= \left( a (r - \delta) y + a (a - 1) \frac{\sigma^2}{2} y \right) dt. \quad (15)$$

If $E^*(dy) = rydt$ then $a$ must satisfy

$$a (r - \delta) + a (a - 1) \frac{\sigma^2}{2} = r. \quad (16)$$

The two solutions are $h_1$ and $h_2$ as given (12.11) and (12.12) which one can verify directly.

**Question 20.11.**

As discussed in the hint, consider a strategy of 1 unit in $Q, -Q \eta_i/(S_i \sigma_i)$ for both $i = 1$ and 2. Let $I_t$ be the amount of money in the risk free asset. The value of the portfolio is

$$V_t = Q_t \left( 1 - \frac{\eta_1}{\sigma_1} - \frac{\eta_2}{\sigma_2} \right) + I_t. \quad (17)$$

The expected change in the value is

$$dV_t = dQ_t - \frac{Q_t \eta_1}{S_1 \sigma_1} dS_1 + \frac{Q_t \eta_2}{S_2 \sigma_2} dS_2 + rI_t dt \quad (18)$$

$$= \left( \left( \alpha_Q - \alpha_1 \frac{\eta_1}{\sigma_1} - \alpha_2 \frac{\eta_2}{\sigma_2} \right) Q + rI \right) dt \quad (19)$$

$$+ (\eta_1 Q - \eta_1 Q) dZ_1 + (\eta_2 Q - \eta_2 Q) dZ_2 \quad (20)$$

$$= \left( \left( \alpha_Q - \alpha_1 \frac{\eta_1}{\sigma_1} - \alpha_2 \frac{\eta_2}{\sigma_2} \right) Q + rI \right) dt. \quad (21)$$

Since this portfolio requires zero investment and there is no risk, the drift and $V_t$ must be zero. Hence

$$\left( \alpha_Q - \alpha_1 \frac{\eta_1}{\sigma_1} - \alpha_2 \frac{\eta_2}{\sigma_2} \right) Q + r \left( 1 - \frac{\eta_1}{\sigma_1} - \frac{\eta_2}{\sigma_2} \right) Q = 0. \quad (22)$$

Rearranging

$$\alpha_Q - r = \frac{\eta_1}{\sigma_1} (\alpha_1 - r) + \frac{\eta_2}{\sigma_2} (\alpha_2 - r). \quad (23)$$
Question 20.12.

We must try to find a position in $S$ and $Q$ that eliminates risk. Let us buy one unit of $S$ and let $\theta$ be the position in $Q$. Let $I_t$ be our bond investment. We have $V_t = S_t + \theta_t Q_t + I_t$ with $V_0 = 0$. Since this strategy must be self financing,

$$dV = (\alpha SS + \theta \alpha Q + r I) \, dt + (\sigma SS - \eta \theta Q) \, dZ \tag{24}$$

hence we will set $\theta = \frac{\sigma SS}{\eta Q}$. This will make our zero cost, self financing strategy riskless. Hence the drift and the value must be zero. Mathematically, if $V_t = 0$ then $I = -S - \frac{\sigma SS}{\eta Q} Q$. The drift being zero implies

$$\alpha SS + \frac{\sigma SS}{\eta Q} \alpha Q - r \left( S + \frac{\sigma SS}{\eta} \right) = 0. \tag{25}$$

Dividing both sides by $S$ and simplifying leads to

$$\alpha Q = r - \frac{\alpha S - r}{\sigma S} \eta. \tag{26}$$

Since $Q$ is negatively related to $Z$, if $S$ has a positive risk premium then $Q$ will negative risk premium.

Question 20.13.

In the following we define $y_t = S_t^a Q_t^b$.

a) From equation (20.38), the (real world) expected value of $y_T$ is $E(y_T) = y_0 e^{mT}$ where

$$m = a (\alpha_S - \delta_S) + b (\alpha_Q - \delta_Q) + \frac{a (a - 1) \sigma_S^2}{2} + \frac{b (b - 1) \sigma_Q^2}{2} + ab \rho \sigma_S \sigma_Q \tag{27}$$

is the real world capital gain. Given a (real world) expected return $\alpha$, the value of the claim is $e^{-\alpha T} E(y_T) = y_0 e^{(m-\alpha)T}$. Using Ito’s lemma and problem 20.11, $\eta_1 = a \sigma_S$ and $\eta_2 = b \sigma_Q$. We then have

$$\alpha = r + a (\alpha_S - r) + b (\alpha_Q - r) \tag{28}$$

and the value of the claim being

$$y_0 e^{(m-\alpha)T} = S_0 Q_0 e^{-rT} e^{hT} \tag{29}$$

where $h = a (r - \delta_S) + b (r - \delta_Q) + \frac{1}{2} a (a - 1) \sigma_S^2 + \frac{1}{2} b (b - 1) \sigma_Q^2 + ab \rho \sigma_S \sigma_Q$. Note this agrees with $e^{-rT} E^\pi (y_T)$.  

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b) The expected return of $y$ is $\alpha$ and the actual expected capital gain is $m$. The lease rate of $y$ would have to be the difference $\delta^* = \alpha - m$ which equals

$$\delta^* = r(1 - a - b) + a\delta_S + b\delta_Q - \frac{1}{2}a(a - 1)\sigma_S^2 - \frac{1}{2}b(b - 1)\sigma_Q^2 - ab\rho\sigma_S\sigma_Q.$$ 

The prepaid forward price must be $y_0e^{-\delta^*T} = y_0e^{(m - \alpha)T}$ which agrees with our previous answer. We can rewrite it in an informative way. The forward price for a security paying $S^a$ is

$$F_{0,T}(S^a) = S^a e^{(r - \delta_S)T + \frac{1}{2}a(a - 1)\sigma_S^2 T}.$$ 

The forward price for $Q^b$ is

$$F_{0,T}(Q^b) = Q^b e^{(r - \delta_Q)T + \frac{1}{2}b(b - 1)\sigma_Q^2 T}.$$ 

Thus, we can rewrite the prepaid forward price as

$$F^p_{0,T}(S^a Q^b) = e^{-rT} F_{0,T}(S^a) F_{0,T}(Q^b) e^{ab\rho\sigma_S\sigma_Q T}. \quad (31)$$

The expression on the right is the product of the forward prices times a factor that accounts for the covariance between the two assets. The discount factor converts it into a prepaid forward price.

**Question 20.14.**

As mentioned in the problem, $\sigma dZ$ appears in both $dS$ and $dQ$. One can think of $dQ$ as an alternative model for the stock (with $dS$ being the standard geometric Brownian motion).

a) If there were no jumps, $dQ$ would also be geometric Brownian motion. Since it has the same risk component, $\sigma dZ$, $\alpha_Q$ must equal $\alpha$. If we thought of $Q$ as another traded asset, this naturally follows from no arbitrage.

b) If $Y_1 > 1$ then there are only positive jumps. We would therefore expect $\alpha_Q < \alpha$ to compensate for this. Mathematically, $dQ/Q - dS/S = (\alpha_Q - \alpha) dt + dq_1$. If a jump occurs, $dq_1 = Y_1 - 1 > 0$; if $\alpha_Q \geq \alpha$ we could buy $Q$ and short $S$. The only risk we have is jump risk but this will always be “good” news for our portfolio. In order to avoid this arbitrage $\alpha_Q$ must be less than $\alpha$.

If we use a weaker assumption $k_1 = E(Y_1 - 1) > 0$ and we assume the returns to $S$ and $Q$ should be the same (this makes sense if we are looking at $Q$ as an alternative model instead of another stock) then we arrive at a similar result. The expected return to $Q$ is $\alpha_Q + \lambda_1 k_1$; setting this equal to $\alpha$ implies $\alpha - \alpha_Q = \lambda_1 k_1 > 0$. 

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c) Let $\alpha^*$ be the expected return of $Q$. Note that $\alpha_Q$ is not the expected return, it is the expected return conditional on no jumps occurring. We have the following relationship,

$$\alpha^* = E \left( \frac{dQ}{Q} \right) / dt = \alpha_Q + k_1 \lambda_1 + k_2 \lambda_2$$

(32)

where $k_i = E (Y_i - 1)$. Hence $\alpha_Q = \alpha^* - k_1 \lambda_1 - k_2 \lambda_2$. If $\alpha^* = \alpha$ (i.e. $Q$ and $S$ have the same expected return) then $\alpha - \alpha_Q = k_1 \lambda_1 + k_2 \lambda_2$. The sign of which could be positive or negative if there are no restriction on $k_1$ and $k_2$. 