Chapter 21
The Black-Scholes Equation

Question 21.1.
If \( V(S,t) = e^{-r(T-t)} \) then the partial derivatives are \( V_S = V_{SS} = 0 \) and \( V_t = rV \). Hence \( V_t + (r - \delta) SV_S + S^2 \sigma^2 V_{SS}/2 = rV \).

Question 21.2.
If \( V(S,t) = AS^{a}e^{\gamma t} \) then \( V_t = \gamma V \), \( V_S = aS^{a-1}e^{\gamma t} = aV/S \), and \( V_{SS} = a(a-1)S^{a-2}e^{\gamma t} = a(a-1)V/S^2 \). Therefore the left hand side of the Black-Scholes equation (21.11) is

\[
V_t + (r - \delta) V_S + V_{SS} S^2 \sigma^2 / 2 - rV = \left( \gamma - r + (r - \delta) a + \frac{\sigma^2}{2} a(a-1) \right) V. \tag{1}
\]

We can rewrite the coefficient of \( V \) as

\[
\gamma + (r - \delta) a + \frac{\sigma^2}{2} a(a-1) = \frac{\sigma^2}{2} a^2 + \left( r - \delta - \frac{\sigma^2}{2} \right) a + \gamma - r. \tag{2}
\]

From the quadratic formula, this has roots

\[
a = \frac{-\left( r - \delta - \frac{\sigma^2}{2} \right)}{\sigma^2} \pm \sqrt{\frac{(r - \delta - \frac{\sigma^2}{2})^2 - 4\frac{\sigma^2 a^2}{2} (\gamma - r)}}. \tag{3}
\]

Simplifying,

\[
a = \left( \frac{1}{2} - \frac{r - \delta}{\sigma^2} \right) \pm \sqrt{\left( \frac{r - \delta}{\sigma^2} - \frac{1}{2} \right)^2 + \frac{2(r - \gamma)}{\sigma^2}}. \tag{4}
\]

Note, for a given \( \gamma \), these are the only values for \( a \) that will satisfy the PDE.

Question 21.3.
If \( V(S,t) = e^{-r(T-t)}S^a \exp \left( (a(r - \delta) + \frac{1}{2}a(a-1)\sigma^2)(T-t) \right) \), we have \( V(S,T) = S_T^a \), hence the boundary condition is satisfied. Note that \( V \) is of the form \( AS^a e^{\gamma t} \), where \( \gamma = r - a(r - \delta) - \frac{1}{2}a(a-1)\sigma^2 \). The previous problem’s result shows \( \gamma \) must solve

\[
a = \left( \frac{1}{2} - \frac{r - \delta}{\sigma^2} \right) \pm \sqrt{\left( \frac{r - \delta}{\sigma^2} - \frac{1}{2} \right)^2 + \frac{2(r - \gamma)}{\sigma^2}}. \tag{5}
\]
Letting \( k = \left( \frac{1}{2} - \frac{r - \delta}{\sigma^2} \right) \), we have to check
\[
a = \pm \sqrt{k^2 + \frac{2(r - \gamma)}{\sigma^2}}. \tag{6}
\]
This is equivalent to checking
\[
k^2 + \frac{2(r - \gamma)}{\sigma^2} \equiv (a - k)^2. \tag{7}
\]
Expanding, this becomes
\[
2\frac{(r - \gamma)}{\sigma^2} \equiv a^2 - 2a \left( \frac{1}{2} - \frac{r - \delta}{\sigma^2} \right). \tag{8}
\]
Solving for \( \gamma \),
\[
\gamma = r - \frac{\sigma^2 a^2}{2} + a \left( \frac{\sigma^2}{2} - (r - \delta) \right) = r - a(r - \delta) - \frac{\sigma^2}{2}a(a - 1) \tag{9}
\]
which is confirmed. One could also do this as a partial derivative exercise.

**Question 21.4.**

Defining \( V(S, t) = K e^{-r(T-t)} + S e^{-\delta(T-t)} \) we have \( V_t = r K e^{-r(T-t)} + \delta S e^{-\delta(T-t)} \), \( V_S = e^{-\delta(T-t)} \) and \( V_{SS} = 0 \). The Black-Scholes equation is satisfied for \( V_t + (r - \delta)V_S S + V_{SS} S^2 \sigma^2 / 2 \) is
\[
r Ke^{-r(T-t)} + \delta S e^{-\delta(T-t)} + (r - \delta) e^{-\delta(T-t)} S \tag{10}
\]
\[
= r \left( Ke^{-r(T-t)} + Se^{-\delta(T-t)} \right) = r V. \tag{11}
\]
This also follows from the result that linear combinations of solutions of the PDE are also solutions. The boundary condition is \( V(S, T) = K + S_T \), i.e. we receive one share and \( K \) dollars. Similarly, a long forward contract with value \( Se^{-\delta(T-t)} - Ke^{-r(T-t)} \) will solve the PDE.

**Question 21.5.**

Let \( V = Se^{-\delta(T-t)} N(d_1) \). Note that \( d_1 \) depends on both \( S \) and \( t \). We have
\[
V_S = e^{-\delta(T-t)} \left( N(d_1) + S \frac{\partial N(d_1)}{\partial S} \right) = e^{-\delta(T-t)} \left( N(d_1) + \frac{N'(d_1)}{\sigma \sqrt{T-t}} \right) \tag{12}
\]
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hence

\[ (r - \delta)S V_S = (r - \delta) V + \frac{(r - \delta)}{\sigma \sqrt{T - t}} e^{-\delta(T - t)} SN'(d_1). \]  \tag{13} 

Similarly,

\[ V_{SS} = e^{-\delta(T - t)} \left( \frac{N'(d_1)}{S \sigma \sqrt{T - t}} + \frac{N''(d_1)}{S \sigma^2 (T - t)} \right) = \frac{e^{-\delta(T - t)} N'(d_1)}{S \sigma \sqrt{T - t}} \left( 1 - \frac{d_1}{\sigma \sqrt{T - t}} \right) \]  \tag{14} 

where we used the fact \( N''(x) = -x N'(x) \). We have

\[ \frac{S^2 \sigma^2 V_{SS}}{2} = \frac{\sigma S e^{-\delta(T - t)} N'(d_1)}{2 \sqrt{T - t}} \left( 1 - \frac{d_1}{\sigma \sqrt{T - t}} \right). \]  \tag{15} 

The partial with respect to \( t \) is

\[ V_t = \delta V + S e^{-\delta(T - t)} N'(d_1) \left( \frac{\ln(S/K)}{2 \sigma (T - t)^{3/2}} - \frac{r - \delta + \sigma^2/2}{2 \sigma (T - t)^{1/2}} \right) \]
\[ = \delta V + \frac{S e^{-\delta(T - t)} N'(d_1)}{2 (T - t)} \left( d_1 - 2 \frac{(r - \delta + \sigma^2/2)}{\sigma} \sqrt{T - t} \right). \]  \tag{16} 

Adding equations (13), (15), and (16), all terms cancel except the \( r V \) term from equation (13), hence \( V_t + (r - \delta) S V_S + S^2 \sigma^2 V_{SS}/2 = r V \) which was to be shown.

**Question 21.6.**

Let \( V(S, t) = e^{-r(T - t)} N(d_2) \); we must show \( V \) solves the PDE \( V_t + (r - \delta) S V_S + S^2 \sigma^2 V_{SS}/2 = r V \). Note that

\[ d_2 = \frac{\ln(S/K)}{\sigma \sqrt{T - t}} + \left( \frac{r - \delta - \sigma^2/2}{\sigma} \right) \sqrt{T - t} \]  \tag{17} 

depends on both \( S \) and \( t \). Beginning with the first term in the PDE,

\[ V_t = r V + e^{-r(T - t)} N'(d_2) \left( \frac{\ln(S/K)}{2 \sigma (T - t)^{3/2}} - \frac{r - \delta - \sigma^2/2}{2 \sigma (T - t)^{1/2}} \right) \]
\[ = r V + \frac{e^{-r(T - t)} N'(d_2)}{2 (T - t)} \left( d_2 - 2 \frac{(r - \delta - \sigma^2/2)}{\sigma} \sqrt{T - t} \right). \]  \tag{18} 

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Since \( V_S = e^{-r(T-t)}N'(d_2) / (S\sigma \sqrt{T-t}) \) the second term in the PDE is

\[
(r - \delta) S V_S = \left( \frac{r - \delta}{\sigma \sqrt{T-t}} \right) e^{-r(T-t)} N'(d_2). \tag{19}
\]

The second partial of \( V \) with respect to \( S \) is

\[
V_{SS} = \frac{e^{-r(T-t)}(N''(d_2) - N'(d_2))}{S^2\sigma^2(T-t)} = \frac{e^{-r(T-t)} N'(d_2)}{S^2\sigma^2(T-t)} \left( d_2 + \sigma \sqrt{T-t} \right) \tag{20}
\]

where we use the property \( N''(x) = -x N'(x) \). The third term in the PDE is therefore

\[
\frac{S^2\sigma^2 V_{SS}}{2} = -\frac{e^{-r(T-t)} N'(d_2)}{2(T-t)} \left( d_2 + \sigma \sqrt{T-t} \right). \tag{21}
\]

Adding equations (18), (19), and (21), all terms cancel except the \( r V \) term in equation (18); i.e. \( V \) satisfies the PDE.

**Question 21.7.**

The two preceding problems, show that each term in the Black-Scholes call option formula satisfies the PDE (these are all or nothing options); since linear combination of solutions to PDEs are also solutions, the Black-Scholes formula solves the PDE. That is

\[
V(S, t) = Se^{-\delta(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2) \tag{22}
\]

The only thing left is to show the boundary condition, \( V(S, T) = \max(S - K, 0) \). The first term is \( SN(d_1) \). As in the text’s discussion of the European call option, at \( t = T \),

\[
N(d_1) = N(d_2) = \begin{cases} 
1 & \text{if } S > K \\
0 & \text{if } S < K 
\end{cases}
\]

hence \( V(S, T) = S - K \) if \( S \geq K \) and \( V(S, T) = 0 \) otherwise.

**Question 21.8.**

These bets are all or nothing options. The cash bets being worth, per dollar, \( e^{-rT}N(d_2) \) if we receive $1 if \( S_T > K \) and \( e^{-rT}N(-d_2) \) if we receive $1 if \( S_T < K \). The stock bets being worth, per share, \( SN(d_1) \) if we receive 1 share if \( S_T > K \) and \( SN(-d_1) \) if we receive 1 share if \( S_T < K \). (Note we are assuming the current time is \( t = 0 \) and the bet is for the stock price \( T \) years from now).
a) By setting $K = Se^{(r-\delta)T}$, $d_2 = -\sigma \sqrt{T}/2$ the value of the bet that the share price will exceed the forward price is $e^{-rT}N(-\sigma \sqrt{T}/2)$. This is always less than the opposite bet, which has value $e^{-rT}N(\sigma \sqrt{T}/2)$.

b) If denominated in cash, we could make the bet fair by setting the strike price equal to $K = Se^{(r-\delta-S\sigma^2)T}$, which is the median (50% of the probability is above this value). This will make $d_2 = 0$ and the bets worth $e^{-rT}/2$ which is not a surprise since the sum of the two bets must be worth $e^{-rT}$. Using $T = 1$, $r = 6\%$, $\sigma = 30\%$, we have $K = 100e^{0.06-3^2/2} = 101.51$.

c) If denominated in shares, we could make the bet fair by setting the strike price equal to $K = Se^{(r-\delta+S\sigma^2)T} = 100e^{0.06+3^2/2} = 111.07$, which is above the forward price. This makes $d_1 = 0$ and the bets worth $S/2 = 50$.

**Question 21.9.**

Let $S = 100$ and $K = 106.184$ which is the forward price. The first bet is worth $V_1 = SN(\sigma \sqrt{T}/2) - e^{-rT}KN(-\sigma \sqrt{T}/2)$ and the second bet is worth $V_2 = KN(\sigma \sqrt{T}/2) - SN(-\sigma \sqrt{T}/2)$. The difference in the values

$$V_1 - V_2 = (S - Ke^{-rT}) \left( N \left( \frac{\sigma \sqrt{T}}{2} \right) + N \left( -\frac{\sigma \sqrt{T}}{2} \right) \right) = S - Ke^{-rT}. \quad (24)$$

Since $K$ is the forward price, $K = Se^{rT}$ which implies $V_1 = V_2$. This is simply put call parity; if the strike price is the forward price, $C - P$ must equal the value of an obligation to buy the asset for the forward price which, by definition is zero. Using the parameters, $\sigma = 30\%$, $r = 6\%$, and $T = 1$, both bets should be worth $11.92$.

**Question 21.10.**

If we purchase one unit of the claim, $-V_S$ shares, and invest $W$ in the risk free bond, our investment is worth $I = V(S, t) - V_SS + W = 0$. By purchasing one claim, we will receive a dividend of $\Gamma dt$ that will be added to $dI$. The change in the investment value is

$$dI = \Gamma dt + V_idt + V_SdS + \frac{\sigma^2S^2V_SdS}{2} - V_SdS - V_S\delta Sdt + rWdt \quad (25)$$

$$= \left( \Gamma + V_i + \frac{1}{2}\sigma^2S^2V_SS - V_S\delta S + rW \right) dt. \quad (26)$$

Since this is risk free and is (initially) a zero investment, both the drift and $I$ must be zero. This implies $W = V_SS - V$ and

$$\Gamma + V_i + \frac{1}{2}\sigma^2S^2V_SS - V_S\delta S + r(V_SS - V) = 0, \quad (27)$$
hence

\[ \Gamma + V_t + \frac{1}{2} \sigma^2 S^2 V_S S + (r - \delta) V_S S = r V. \]  

(28)

Note that if we assume \( \Gamma \) is a continuous yield of the claim (rather than a $ per unit rate), the first term would be \( \Gamma V \) rather than \( \Gamma \).

**Question 21.11.**

Using the notation from Proposition 21.1, \( \eta = .02 + 2 (.2) .3 (.5) = .08, \delta^* = .06 - 2 (.06 - .01) - .5^2 = -.29 \). The function \( V \) is the prepaid forward price of \( S \), \( S_0 e^{-\eta T} \). The value of the claim is

\[ 90^2 e^{(.06+.29)2}50e^{-08(2)} = 694, 983. \]  

(29)

Using proposition 20.4, the value should be

\[ S_0 e^{-\delta T} \left( Q_0 e^{b(r-\delta Q) + .5b(b-1)\sigma^2 \sigma^2} \right) e^{bp_0 \sigma \sigma T} \]  

(30)

which equals

\[ 50e^{-04} \left( 90^2 e^{(1+.5^2)2} \right) e^{-12} = 694, 983. \]  

(31)

**Question 21.12.**

Setting \( b = -1 \) and using Proposition 21.1, we change the dividend yield of \( S \) to \( \eta = .02 - .2 (.3) (.5) = -.01 \). The prepaid forward price, i.e. \( V \) in equation (21.35), is \( S_0 e^{-\eta T} \). Letting \( \delta^* = .06 + (.06 - .01) - .5^2 = -.14 \), we have the value of the claim being

\[ \frac{1}{90} e^{-2(2)} \left( 50e^{.01(2)} \right) = 0.8455. \]  

(32)

Using Proposition 20.4, the claim should be worth

\[ S_0 e^{-\delta T} \left( Q_0 e^{b(r-\delta Q) + .5b(b-1)\sigma^2 \sigma^2} \right) e^{bp_0 \sigma \sigma T} \]  

(33)

which equals

\[ 50 e^{-04} \left( 90^{-1} e^{-05+.5^2} \right) e^{03(2)} = 0.8455. \]  

(34)

Note that Proposition 20.4 derives the forward price; upon discounting, the forward price of \( S \) becomes \( S_0 e^{-\delta T} \) and the forward price of \( Q^b \) terms does not get discounted.

Let \( P(Q, S, 0) \) be the current \((t = 0)\) no arbitrage value of the claim that pays \( [Q_T - F_{0,T}] \times \max(0, S_T - K) \). Since \( F_{0,T}(Q) = Qe^{(r - \delta Q)T} \) (a “known” number)

\[
P(Q, S, 0) = Qe^{(r - \delta Q)T} V(S, K, \sigma_S, r, T, \delta - \rho \sigma \sigma_S) - Qe^{(r - \delta Q)T} V(S, K, \sigma_S, r, T, \delta).
\] (35)

where \( V(\cdot) \) is the Black-Scholes call option formula; note that there is a different dividend yield in the two equations. We immediately see that, since \( \rho < 0 \), the first option will be worth less than the second and we shouldn’t accept this offer. Intuitively, since \( \ln(S) \) and \( \ln(Q) \) are negatively correlated, when \( Q_T > F_{0,T}(Q) \), the call option is more likely to be out of the money. Using \( K = 50 \), the claim will be worth

\[
90e^{(0.06 - 0.01)^2} (7.98 - 10.39) = -239.71.
\] (36)


Using Proposition 21.1, since \( b = 1 \), the insurance payoff should be worth

\[
Qe^{(r - \delta Q)T} V(S, K, \sigma_S, r, T, \delta - \rho \sigma \sigma_S)
\] (37)

hence we should use a dividend yield of \( .02 + .2(.3)(.5) = .05 \) making the put relatively more valuable. For \( K = 50, V = 7.09 \) hence the insurance is worth \( 90e^{(0.06 - 0.01)^2} (7.09) = 705.21 \). If we wanted to insure \( 90e^{(0.06 - 0.01)^2} = 99.465 \) units, it would cost \( 90e^{(0.06 - 0.01)^2} (6.05) = 601.77 \). This is intuitive since \( \ln(S) \) and \( \ln(Q) \) are negatively correlated. When \( Q \) is high, \( S \) is more likely to be low making the insurance payout larger (the holder has the right to sell \textit{more} units for \( K \)).