Forward premium and the annualized forward premium
An example with forward contracts and arbitrage

10.1. Introduction. The forward premium is meant to reflect the ratio of the current forward price on a stock to the stock price. The annualized forward premium (rate) also normalizes the forward premium using the length of time to the delivery date of the forward. Both measures are useful to try to infer the stock price in markets that do not have frequent trades in the underlying asset (so that the traders are not confident in the stock prices that were last observed a relatively long time ago).

10.2. Definition. As usual, let $F_{0,T}(S)$ denote the forward price for the delivery of asset $S$ at time $T$. Then, the forward premium is defined as

$$\frac{F_{0,T}(S)}{S(0)}.$$ 

The annualized forward premium is defined as

$$\frac{1}{T} \ln \left( \frac{F_{0,T}(S)}{S(0)} \right).$$

10.3. Interpretation. Let us temporarily write $\alpha(S)$ for the annualized forward premium of the asset $S$. Then, for every $T$, we have

$$\alpha(S) = \frac{1}{T} \ln \left( \frac{F_{0,T}(S)}{S(0)} \right) \Rightarrow F_{0,T}(S) = S(0)e^{\alpha(S)T}.$$ 

Let us look at the simple case of an asset which pays continuous dividends at the rate $\delta$. We still denote the continuously compounded interest rate by $r$. Then, the above equality gives us

$$S(0)e^{(r-\delta)T} = S(0)e^{\alpha(S)T} \Rightarrow r - \delta = \alpha(S).$$

So, in this case the annualized forward premium rate reflects “mean appreciation” of the stock itself.

10.4. Problem. The current price of a stock is $S(0) = $125 per share. Let the stock pay continuous dividends at the continuous dividend rate $\delta$. Assume that the continuously compounded interest rate equals $r = 0.3$. The prepaid forward price for delivery of the above stock in two years is $83.79. Calculate the annualized forward premium (rate).
Solution. Based on the above discussion, we conclude that the answer equals
\[ r - \delta = 0.3 - \delta. \]

We use the prepaid forward price do calculate the \( \delta \).
\[
F_{0,T}^P(S) = S(0)e^{-\delta T} \implies \delta = -\frac{1}{T} \ln \left( \frac{F_{0,T}^P(S)}{S(0)} \right) = -\frac{1}{2} \ln \left( \frac{83.79}{125} \right) \approx 0.2.
\]

So, the final answer is about 0.1.

10.5. **Forwards and arbitrage: An example.** Suppose that the current price of a dividend-paying stock equals $1,000. Let \( r = 0.25 \) and \( \delta = 0.15 \). You notice that a forward price for delivery of this stock in two-years equals \( F = 1,200 \). You suspect that this forward price creates an arbitrage opportunity. The reason for this suspicion is that the forward price based on the initial stock price, \( r \) and \( \delta \) equals
\[
F_{0,T}(S) = S(0)e^{(r-\delta)T} = 1000e^{(0.25-0.15)\cdot2} \approx 1,221.4 > F = 1,200.
\]

The conclusion is that the observed forward price is “too low”. One way to exploit this arbitrage opportunity would be to do the following:

1. engage in the **long** forward contract,
2. **short-sell** \( e^{-\delta T} \) shares of stock,
3. invest the proceeds from the short sale at the risk-free rate.

So, the initial cost of this portfolio is zero.

During the time period \((0,T]\), all of the continuously paid dividends are automatically reinvested in the asset \( S \). So, at the end, one share of stock needs to be returned. Thus, at time \(-T\), the payoff is
\[
(S(T) - F) - S(T) + e^{(r-\delta)T}S(0) = 21.4 > 0.
\]

The portfolio we constructed is, indeed, an arbitrage portfolio.