2.1. TRUE/FALSE QUESTIONS.

Problem 2.1. (2 pts) All else being equal, American put options are at least as valuable as European put options.

Solution: TRUE

Problem 2.2. (2 pts) An American call option on a non-dividend paying stock should never be exercised early. More precisely, it is never more profitable to early exercise an American call option on a stock which pays no dividends prior to the expiry date of the option.

Solution: TRUE

Problem 2.3. (2 pts) Let $V_A(0,T)$ denote the price at time 0 of an American option with expiration date $T$. Then, we always have

$$V_A(0,T) \leq V_A(0,2T).$$

Solution: TRUE

2.2. FREE-RESPONSE PROBLEMS. Please, explain carefully all your statements and assumptions. Numerical results or single-word answers without an explanation (even if they’re correct) are worth 0 points.

Problem 2.4. (10 points) Assume that a stock pays no dividends. Its initial price is given to be $2.

Consider two European-style derivative securities on the above stock, both with the exercise date in one year. They have the following payoffs:

I: $V_I(T) = (5S(T) - 10)_+$
II: $V_{II}(T) = (10 - 5S(T))_+$

with $T = 1$. It is observed that the price of derivative $I$ at time−0 equals $V_I(0) = 2$.

Given that the continuously compounded risk-free interest rate equals 0.05, what is the no-arbitrage time−0 price of derivative $II$?
Solution: Notice that
\[
V_I(T) = (5S(T) - 10)_+ = 5V_C(T) \\
V_{II}(T) = (10 - 5S(T))_+ = 5V_P(T),
\]
where \( V_C(T) \) and \( V_P(T) \) denote the payoffs of a 2-strike call and a 2-strike put, respectively, with the same underlying and exercise date as the two derivative securities described in the problem. We conclude that \( V_I(0) = 5V_C(0) \) and \( V_{II}(0) = 5V_P(0) \), in our usual notation.

On the other hand, put-call parity gives us
\[
V_C(0) - V_P(0) = S(0) - Ke^{-rT} \\
\Rightarrow \quad V_P(0) = V_C(0) - S(0) + Ke^{-rT}.
\]
Finally,
\[
V_{II}(0) = 5(V_C(0) - S(0) + Ke^{-rT}) = 5V_C(0) - 5S(0) + 5Ke^{-rT} = 2 - 10 + 10e^{-0.05} \approx 1.51.
\]

Problem 2.5. (10 pts) A certain common stock is priced at $42.00 per share. Assume that the continuously compounded interest rate is \( r = 10.00\% \) per annum. Consider a $50–strike European call, maturing in 3 years which currently sells for $10.80. What is the price of the corresponding 3-year, $50–strike European put option?

Solution: Due to put-call parity, we must have
\[
V_P(0) = V_C(0) + e^{-rT}K - S(0) = 10.80 + e^{-0.30} \cdot 50 - 42.00 \approx 5.84.
\]

Problem 2.6. (30 points) Consider a two-period binomial model with \( S(0) = $50, u = 2 \) and \( d = 0.5 \).

(i) (5 points) Draw the binomial tree modeling the future evolution of this stock price with the given \( u \) and \( d \).

Your goal is to price an at-the-money European call option with two periods to maturity under the following assumptions:
the underlying stock does not pay any dividends,
the effective risk-free interest rate per period equals \( i = 25\% \).

(ii) (5 pts) Find the risk-neutral probability.

(iii) (8 pts) Find the fair price of the call using the risk-neutral pricing formula.

(iv) (12 pts) Find the \( \Delta \) which should be used at every node in the tree in order to form a replicating portfolio. More precisely, in the notation used in class, calculate \( \Delta_u, \Delta_d \) and \( \Delta_0 \).
Solution: The binomial tree looks like this:

The risk-neutral probability of the stock price going up in a single period is simply:

\[ p^* = \frac{(1 + i) - d}{u - d} = 0.5. \]

We proceed backwards through the inner nodes of the tree. At the “up” node, we have that the value of the call is

\[ V_u = \frac{1}{1.25^2} [150 + 0] = 60. \]

On the other hand, the remaining two final payoffs are both zero, which yields that the value of the call at the “down” node equals

\[ V_d = 0. \]

We have now reduced the pricing problem to a one-step binomial tree. The usual calculation gives us that the fair price of the above call is

\[ V_C(0) = \frac{1}{1.25^2} [60 + 0] = 24. \]

With the usual notation, we have

\[ \Delta_u = \frac{150 - 0}{1.5 \cdot 100} = 1; \quad \Delta_d = 0; \quad \Delta_0 = \frac{60 - 0}{1.5 \cdot 50} = \frac{4}{5}. \]

Problem 2.7. (20 points) The current price of a share of stock \( S \) is $100. The stock is assumed to be paying a continuous dividend with the dividend yield of 0.04.

Assume that the continuously compounded interest rate equals 0.05
Consider the following European gap options with the same exercise date in one year and the same underlying asset \( S \).

I Gap call with strike price 100 and trigger price 100
II Gap put with strike price 100 and trigger price 100
III Gap call with strike price 100 and trigger price 110
IV Gap call with strike price 110 and trigger price 100
V Gap call with strike price 100 and trigger price 80.

Which one of the above options has the highest price?

**Solution:** Let us try compare the prices of options I and II, first. Since for the both of them the trigger and the strike prices are the same, we are in fact dealing with just plain vanilla options. The “regular” put-call parity applies, and in our usual notation, we have

\[
V_I(0) - V_{II}(0) = F_{0,T}^P(S) - 100e^{-rT} = 100e^{-0.04} - 100e^{-0.05} = 100(e^{-0.04} - e^{-0.05}) > 0
\]

Option III has a lower price than option I since the payoff curve for option I dominates the payoff of option III.

Using the same type of comparison, we see that the value of option I is greater than the value of option IV (again, the payoff curve for option I is always above or at the same level as the payoff curve for option IV.)
Option I has the higher price than option V (again, its payoff curve is always above or at the same level as the payoff curve for option V). So, the price for option I is higher than the price of option V.

We conclude that the option with the highest price of the ones offered is option I.

2.3. MULTIPLE CHOICE QUESTIONS. Please note your answers on the front page.

Problem 2.8. (5 points) Source: Problem #2 from the Sample FM(DM) questions.
You are given the following information:
(1) The current price to buy one share of XYZ stock is 500.
(2) The stock does not pay dividends.
(3) The risk-free interest rate, compounded continuously, is 6%.
(4) A European call option on one share of XYZ stock with a strike price of \( K \) that expires in one year costs $66.59.
(5) A European put option on one share of XYZ stock with a strike price of \( K \) that expires in one year costs $18.64.

Determine the strike price \( K \).
(a) $449
(b) $452
(c) $480
Solution: (c)
This problem is a simple application of put-call parity. In our usual notation,
\[ V_C(0) - V_P(0) = S(0) - e^{-rT}K \]
\[ \Rightarrow K = e^{rT}(S(0) - V_C(0) + V_P(0)) = e^{0.06\cdot1}(500 - 66.59 + 18.64) = 480. \]

Problem 2.9. The price today of a common stock is $100 per share. You are given that:
1. Dividends in equal amounts are to be paid in exactly 2 months and then again in 4 months.
2. A European call on the above stock with strike \( K = $100 \) and the exercise date in six months sells for $7.42.
3. A European put on the above stock with strike \( K = $100 \) and the exercise date in six months sells for $8.90.
4. The continuously-compounded risk-free interest rate equals 0.05.
Calculate the amount of each dividend.
(a) About 5
(b) About 4
(c) About 3
(d) About 2
(e) None of the above

Solution: (d)
In addition to our usual notation, we introduce \( D \) to stand for the amount of each dividend payment. Then, the put-call parity reads as
\[ V_C(0) - V_P(0) = S(0) - De^{-rt_1} - De^{-rt_2} - Ke^{-rT} \]
with \( t_1 = 1/6 \) and \( t_2 = 1/3 \). Solving for \( D \) above, we get
\[ D = \frac{S(0) - Ke^{-rT} - V_C(0) + V_P(0)}{e^{-rt_1} + e^{-rt_2}} = \frac{100 - 100e^{-0.05\cdot(1/2)} - 7.42 + 8.9}{e^{-0.05\cdot(1/6)} + e^{-0.05\cdot(1/3)}} \approx 2. \]

Problem 2.10. A certain common stock is priced at $36.50 per share. The company just paid its $0.50 quarterly dividend. Assume that the interest rate is \( r = 6.0\% \). Consider a $35 strike European call, maturing in 6 months which currently sells for $3.20. What is the price of the corresponding 6-month, $35 strike put option?
(a) $1.20
(b) $1.69
(c) $2.04
(d) $2.38
(e) None of the above.
Solution: (b)

Due to put-call parity, we must have

\[ V_P(K = 35, T = 0.5) = V_C(K = 35, T = 0.5) + e^{-rT}K - S(0) + PV_0T(Div) \]

\[ = 3.20 + e^{-0.03} \cdot 35 - 36.50 + e^{-0.06-0.25} \cdot 0.50 + e^{-0.06-0.5} \cdot 0.50 \]

\[ = -33.230 + 0.97 \cdot 35 + 0.98 \cdot 0.50 + 0.97 \cdot 0.50 \]

\[ = -33.230 + 34.435 + 0.49 = 1.695. \]

Problem 2.11. Let \( K_1 = 50, K_2 = 60 \) and \( K_3 = 65 \) be the strikes of three European call options on the same underlying asset and with the same expiration date. Let \( V_C(K_i) \) denote the price at time 0 of the option with strike \( K_i \) for \( i = 1, 2, 3 \). We are given that \( V_C(K_1) = 12 \) and \( V_C(K_3) = 5 \). What is the maximum possible value of \( V_C(K_2) \) which still does not violate the convexity property of call option prices?

(a) About $16/3

(b) About $7

(c) About $22/3

(d) About $8

(e) None of the above.

Solution: (c)

From the given parameters, we see that \( K_2 = \lambda K_1 + (1 - \lambda)K_3 \) for \( \lambda = 1/3 \). So, we must have

\[ \lambda V_C(K_1) + (1 - \lambda)V_C(K_3) \geq V_C(K_2). \]

The extreme case, i.e., the equality is obtained above for

\[ V_C(K_2) = (1/3) \cdot 12 + (2/3) \cdot 5 = 22/3. \]

Problem 2.12. (5 pts) Which of the following models always satisfies the no-arbitrage condition for the construction of the binomial stock-price tree, regardless of the choice of parameters \( r, \delta, h, \sigma, T, n, S(0) \)?

(a) The forward binomial tree.

(b) The Cox-Ross-Rubinstein model.

(c) The lognormal tree.

(d) Any of the three models.

(e) None of the three models.

Solution: (a)

It was discussed in class that the forward binomial tree always satisfies the no-arbitrage condition. So, the only two acceptable choices above are (a) and (d).

To discard the Cox-Ross-Rubinstein (CRR) model as always being arbitrage free, consider the situation in which \( \sigma = 0.1, r = 0.3, \) and \( \delta = 0 \). In the graph below, the red line in the is the value of the \( u \) in the CRR model while the blue line stands for the \( e^{(r-\delta)h} \) both as functions of the length of a single period \( h \).
As you can see, if we choose too small a value for $h$, the up factor $u$ falls below $e^{rh}$ so that the no-arbitrage condition is violated. This was discussed in class. Now you have an actual choice of parameters which can backfire in a manner of speaking. This means that the offered choice (d) is no longer acceptable.

**Problem 2.13.** (5 points) An investor acquires a call bull spread consisting of the call with strike $K_1 = 100$ and $K_2 = 110$ and with expiration in one year. The initial price of the 100–strike call option equals $11.34, while the price of the 110–strike option equals $7.74. At expiration, it turns out that the stock price equals $105$. Given a continuously compound annual interest rate of 5.0%, what is the profit to the investor?

(a) $3.78$ loss  
(b) $1.22$ loss  
(c) $1.22$ gain  
(d) $5$ gain  
(e) None of the above.

**Solution:** (c)
The total initial cost of establishing the investor’s position is $11.34 - 7.74 = 3.60$. The future value of this amount at expiration is $3.60e^{0.05} = 3.78$. The payoff at expiration is

$$(S(T) - 100)_+ - (S(T) - 110)_+ = (105 - 100)_+ - (105 - 110)_+ = 5.$$  

So, the profit is $5 - 3.78 = 1.22$.

**Problem 2.14.** (5 points) Assume that the continuously compounded interest rate equals 0.05.

Stock $S$ has the current price of $S(0) = 100$ and pays continuous dividends at the rate $\delta_S$. Stock $Q$ has the current price of $Q(0) = 100$ and it pays continuous dividends at the rate of 0.02.

An exchange option gives its holder the right to give up one share of stock $Q$ for a share of stock $S$ in exactly one year. The price of this option is $10.12$.

Another exchange option gives its holder the right to give up one share of stock $S$ for a share of stock $Q$ in exactly one year. The price of this option is $13.02$.

Find $\delta_S$. 

(a) 0  
(b) 0.02  
(c) 0.05  
(d) 0.07  
(e) None of the above.

**Solution: (c)**

By the generalized put-call parity,

\[ V_{EC}(Q(0), S(0), 0) + F_{0,T}^P(S) = V_{EC}(S(0), Q(0), 0) + F_{0,T}^P(Q). \]

So,

\[ 13.02 + 100e^{-\delta_S} = 10.12 + 100e^{0.02}. \]

We get

\[ \delta_S = -\ln \left( \frac{13.02 - 10.12 + 100e^{-0.02}}{100} \right) = 0.05. \]