3.1. MULTIPLE CHOICE QUESTIONS.

Problem 3.1. (5 pts) For a two-period binomial model, you are given that:

1. each period is one year;
2. the current price of a non-dividend paying stock $S$ is $S(0) = 20$;
3. $u = 1.3$, with $u$ as in the standard notation for the binomial model;
4. $d = 0.9$, with $d$ as in the standard notation for the binomial model;
5. the continuously compounded risk-free interest rate is $r = 0.05$.

Find the price of an American call option on the stock $S$ with $T = 2$ and the strike price $K = 22$.

(a) $1$
(b) $2$
(c) $3$
(d) $4$
(e) None of the above
Problem 3.4. (2 points) Let the continuously compounded interest rate be denoted by \( r \). Consider a futures contract for delivery at time \(-T\) of a market index with the continuous dividend yield \( \delta \). As a function of time, the price of this contract at time \( t \) is denoted by \( F_{t,T} \). Denote the time--\( t \) price of a European call on the futures contract with strike \( K \) and exercise date \( T^* < T \) by \( V_C(t) \), and denote the time--\( t \) price of a European put on the same futures contract with the same strike price and the same exercise date by \( V_P(t) \). Then, the following equality is always true
\[
V_C(t) - V_P(t) = F_{t,T}e^{-\delta(T-t)} - Ke^{-rT}.
\]

Problem 3.5. (2 points) In our usual notation, let \( S(0) = 40, r = 0.08, \sigma = 0.3, \delta = 0 \). You need to construct a 2--period forward binomial tree for the above stock with every period in the tree of length \( h = 0.5 \). Then, \( u > 1.45 \).

Problem 3.6. (2 points) In the usual notation for the binomial asset pricing model, we always have
\[
1 < d < u.
\]

Problem 3.7. (2 pts) In the setting of the binomial asset pricing model, with \( i \) denoting the effective interest rate per period and assuming that the underlying asset pays no dividends: If
\[
d < u \leq 1 + i
\]
then there is no possibility for arbitrage.

Problem 3.8. (2 pts) Suppose that the European options with the same maturity and the same underlying assets have the following prices:

1. a 50--strike call costs $9;
2. a 55--strike call costs $10;

Then, some of the monotonicity conditions for no-arbitrage are violated by the above premiums.

3.3. FREE RESPONSE PROBLEMS.

Problem 3.9. (10 points) An investor wants to hold 200 euros two years from today. The spot exchange rate is $1.31 per euro. If the euro denominated annual interest rate is 3.0% what is the price of a currency prepaid forward?

Problem 3.10. (20 points) Consider a one-period binomial model with \( S(0) = 105, S_u = 130 \) and \( S_d = 80 \). Your goal is to determine if there is an arbitrage opportunity in a market in which a European call option on \( S \) with strike of \( K = 120 \) and exercise date \( T = 1 \) year is sold for $5. Assume that the continuously compounded risk-free interest rate equals \( r = 10% \).

If you believe that there is an arbitrage opportunity, describe the arbitrage portfolio and show that it is, indeed, an arbitrage portfolio. If you believe that there is no arbitrage opportunity, explain your reasoning.
Problem 3.11. (10 points) Let \( S(0) = 40, r = 0.08, \sigma = 0.3, \delta = 0 \). You need to find the up and down factors in a 2-period \textbf{forward} binomial tree modeling the price of this stock during the following year.

(a) (4 pts) What are \( u \) and \( d \)?

(b) (6 pts) What is the risk-neutral probability of the stock price going up in a single period?

Problem 3.12. (10 points)

(i) (5 pts) Calculate the price of a long butterfly spread using the following call options:

1. a £3,925–strike call on the FTSE100 index which is being sold for £713.07;
2. a £4,325–strike call on the FTSE100 index which is being sold for £496.46;
3. a £4,725–strike call on the FTSE100 index which is being sold for £333.96.

(ii) (5 pts) Assume that the index pays no dividends. Use the put-call parity to derive the price of the corresponding butterfly spread in terms of the prices of put options analogous to the call options listed above.

Problem 3.13. (25 points) Consider a two-period binomial model for a non-dividend paying asset \( S \) with \( S(0) = 50 \) and \( u = 1/d = 2 \). Let \( i = 0.25 \) denote the effective interest rate per period. You need to price a European put option on \( S \) which expires at the end of the two periods and has the strike \( K = 70 \).

(i) (10 pts) Find the values of the given option at all the nodes in the binomial tree. In particular, find the fair price at time 0 of this option.

(ii) (10 pts) Find the number of shares \( \Delta \) one needs to invest in at every node in the tree in order to replicate the option.

(iii) (5 pts) If the option were American, would there be early exercise?
Problem 3.14. (18 points) Let $S(0) = $80, $K = $95, $r = 8\%$, $T = 1$ and $\delta = 0$. Assuming that $u = 1.3$ and $d = 0.8$, construct a two-period binomial tree for a call option. Provide all the entries in the following table:

<table>
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<tr>
<th>Period</th>
<th>Stock movement</th>
<th>Option premium</th>
<th>$\Delta$</th>
<th>B</th>
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</tr>
<tr>
<td>1</td>
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