Understanding Risk-Neutral Probability
Risk-Neutral Pricing Revisited

- We can interpret the terms \((e^{(r-\delta)h} - d)/(u - d)\) and 
  \((u - e^{(r-\delta)h})/(u - d)\) as probabilities
- Let
  \[
p^* = \frac{e^{(r-\delta)h} - d}{u - d}
\]
  \[(10.5)\]
- Then equation (10.3) can then be written as
  \[
  C = e^{-rh} \left[ p^* C_u + (1 - p^*)C_d \right]
  \]
  \[(10.6)\]
  - Where \(p^*\) is the \textbf{risk-neutral probability} of an increase in the stock price
Understanding Risk-Neutral Pricing

- A **risk-neutral** investor is indifferent between a sure thing and a risky bet with an expected payoff equal to the value of the sure thing (compare to risk-averse)

- $p^*$ is the risk-neutral probability that the stock price will go up
Understanding Risk-Neutral Pricing (Cont’d)

• The option pricing formula can be said to price options as *if* investors are risk-neutral
  – Note that we are not assuming that investors are actually risk-neutral, and that risky assets are actually expected to earn the risk-free rate of return
Pricing an Option Using Real Probabilities

• Is option pricing consistent with standard discounted cash flow calculations?

  Yes.

  However, discounted cash flow is not used in practice to price options
Pricing an Option Using Real Probabilities (cont’d)

• Suppose that the continuously compounded expected return on the stock is \( \alpha \) and that the stock does not pay dividends

• If \( p \) is the true probability of the stock going up, \( p \) must be consistent with \( u \), \( d \), and \( \alpha \)

\[
p u S + (1 - p) d S = e^{\alpha h} S
\]

(11.3)

• Solving for \( p \) gives us

\[
p = \frac{e^{\alpha h} - d}{u - d}
\]

(11.4)
Pricing an Option Using Real Probabilities (cont’d)

• Using $p$, the actual expected payoff to the option one period hence is

$$pC_u + (1 - p)C_d = \frac{e^{\alpha h} - d}{u - d} C_u + \frac{u - e^{\alpha h}}{u - d} C_d$$ (11.5)

• At what rate do we discount this expected payoff?
  – It is not correct to discount the option at the expected return on the stock, $\alpha$, because the option is equivalent to a leveraged investment in the stock and hence is riskier than the stock
Pricing an Option Using Real Probabilities (cont’d)

- Denote the appropriate per-period discount rate for the option as $\gamma$

- Since an option is equivalent to holding a portfolio consisting of $\Delta$ shares of stock and $B$ bonds, the expected return on this portfolio is

$$e^{\gamma h} = \frac{S\Delta}{S\Delta + B} e^{\alpha h} + \frac{B}{S\Delta + B} e^{r h}$$

(11.6)
Pricing an Option Using Real Probabilities (cont’d)

- We can now compute the option price as the expected option payoff, discounted at the appropriate discount rate, given by equation (11.6). This gives

\[ C = e^{-\gamma h} \left[ \frac{e^{\alpha h} - d}{u - d} C_u + \frac{u - e^{\alpha h}}{u - d} C_d \right] \]

(11.7)
Pricing an Option Using Real Probabilities (cont’d)

• It turns out that this gives us the same option price as performing the risk-neutral calculation
  – Note that it does not matter whether we have the “correct” value of $\alpha$ to start with
  – Any consistent pair of $\alpha$ and $\gamma$ will give the same option price
  – Risk-neutral pricing is valuable because setting $\alpha = r$ results in the simplest pricing procedure.