Lecture 1 (Chapter 1, Section 1.2)

The language of sets

Definition (Cantor): A set is a collection of elements (or objects)

\[ \boxed{\text{A set}} \]

- \( C \) the set of all countries in the EU. France is an object in \( C \).
- \( \{2, 4, 6\} \) is the set of numbers with elements 2, 4, 6.

Notation: A set is specified using curly brackets \( \{ \ldots \} \). It is called the set-roster notation.

\[ \boxed{\text{Notation}} \]

\( \boxed{\text{A set}} \)

- \( A = \{1, 2, 3\} \), \( B = \{3, 1, 2\} \), \( C = \{1, 1, 2, 3, 3, 3\} \)

What are the elements of \( A, B, C \)? How are \( A, B, C \) related?

Solution: elements: 1, 2, 3; \( A = B = C \). 3 different ways to represent the same set.

- How many elements are in the set \( A = \{1, 1\} \)? \( B = \{1, 2, 3\} \)?

Solution: \( A \) has only 1 element, namely the number 1.

\( B \) has 2 elements: the number 1, and the set \( \{1, 3\} \) with the number 1 as unique element.

\( \boxed{\text{Careful}} \): \( 1 \neq \{1, 3\} \) !!! 1 is a number, \( \{1, 3\} \) a set.

Some special sets

Some sets of numbers are used very frequently and have a special notation.
Set
\[\mathbb{N} = \{ 0, 1, 2, 3, \ldots \}\] natural numbers
\[\mathbb{Z} = \{ \ldots, -2, -1, 0, 1, 2, \ldots \}\] integers
\[\mathbb{Q} = \text{all numbers } q \text{ which can be written as fractions } q = \frac{a}{b}, \text{ with } a, b \in \mathbb{Z}\]
\[\mathbb{R} = \mathbb{Q} + \text{all irrational numbers}\]

Further notations:
- Adding a superscript \( + \) or \( - \) or "nonneg" indicates that only the positive, negative, or nonnegative elements of the set are included.
- The symbol \( \in \) means "is an element of".
- The symbol \( \notin \) means "is not an element of".

Let \( S \) be a set, \( P(x) \) a property that elements of \( S \) have to satisfy. We can define the following new set:
\[\{ x \in S \mid P(x) \}\] "set of all elements \( x \) in \( S \) such that property \( P(x) \) is true".
Describe the following sets (and draw a picture).

a) \( \{ x \in \mathbb{R} \mid -2 < x < 5 \} = A \)

b) \( \{ x \in \mathbb{Z} \mid -2 < x < 5 \} = B \)

c) \( \{ x \in \mathbb{Z}^+ \mid -2 < x < 5 \} = C \)

d) \( \{ x \in \mathbb{R} \mid x^2 + 1 = 0 \} = D \)

Solution:

a) \( A \) is the set of real numbers such that \(-2 < x < 5\).

b) \( B \) is the set of integers between \(-2\) and \(5\) (not included).

\[ B = \{-1, 0, 1, 2, 3, 4\} \]

\[ -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \]

c) \( C \) is the set of positive integers between \(-2\) and \(5\).

\[ C = \{1, 2, 3, 4\} \]

\[ 1 \quad 2 \quad 3 \quad 4 \]

d) \( D \) is the set of real numbers such that \(x^2 + 1 = 0\) \(\iff\) \(x^2 = -1\). There are no such integers \(\Rightarrow\) \(D = \emptyset\), the empty set.

Notation: We use the symbol \(\emptyset\) to describe the empty set.

Careful: It is different from the number \(0\)!!

Subsets:

Definition: Let \( A \) and \( B \) be two sets. \( A \) is called a subset of \( B \) if and only if every element of \( A \) is also an element of \( B \).

Notation: \( A \subseteq B \) (or sometimes \( A \subsetneq B \)).
\[ \delta x : \mathbb{Q} \subseteq \mathbb{R}, \ \mathbb{Z} \ni 1, 2, 3 \subseteq \mathbb{Z}. \]

* A \not\subseteq B means "there is at least one element x, "A is not a subset of B" such that x \in A, x \notin B."

**Definition:** let A, B be sets. A is a proper subset of B if and only if every element of A is in B but there is at least one element of B not in A.

\[ \delta x : A = \mathbb{Z}^+, B = \{ n \in \mathbb{Z} \mid 0 \leq n \leq 100 \}, C = \{100, 200, 300, 400\} \]

Say if the following statements are true:

a) A \subseteq B
b) C is a proper subset of A
c) C and B have at least one element in common.
d) C \subseteq B, d') C \subseteq C

**Solutions:**

a) \[ \mathbb{Z}^+ = \{1, 2, 3, 4, \ldots \}, B = \{0, 1, 2, \ldots, 100\}. \]
   Consequently: \(0 \in B\) but \(0 \notin \mathbb{Z}^+ \rightarrow \) \(B \not\subseteq \mathbb{Z}^+\) *False*

b) *True* : \(\{100, 200, 300, 400\} \subseteq \mathbb{Z}^+\), but for example \(99 \in \mathbb{Z}^+\) but \(99 \notin A\).

c) *True* : 100 \(\in C\) and 100 \(\in B\) (it is by the way the only one)

d) *False* : 200 \(\in C\) but 200 \(\notin B\) for example.

d') *True* : trivially all elements in C are in C.

* Don't confuse "\(\in\)" and "\(\subseteq\)" : "\(\subseteq\)" is a relation between 2 sets, "\(\in\)" between a number/object and a set.
$\forall x$ : Is the notation right?

a) $2 \in \{1, 2, 3\}$, b) $\{2\} \in \{1, 2, 3\}$, c) $2 \leq \{1, 2\}$

d) $\{2\} \subseteq \{1, 2, 3\}$, e) $\{2\} \subseteq \{1\}, \{2\} \subseteq \{1, 2\}$

c) $\{2\} \subseteq \{1, 2\}$

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Solution:

a) True: $1, 2, 3$ are the elements of $\{1, 2, 3\}$ $\Rightarrow$ $2$ is an element in the set.

b) False: $\{2\} \neq 2$, $\{2\}$ is not one of the elements $1, 2, 3$.

c) False: $2$ is a number, not a set.

d) True: $2$ is the only element in $\{2\}$ and $2 \in \{1, 2, 3\}$ as well.

e) False: The elements of $\{2, 1\}, \{2\}$ are $2, 1, 2$ but $2 \neq 1, 3$ and $2 \notin 1, 2$.

b) True: $\{2\}$ is an element of $\{2, 1\}, \{2\}$.

Some additional notations:

- " $\Rightarrow$ " means "it follows ..."
- " $\iff$ " means "is equivalent to"
- " $\exists$ " means "there exists"
- " $\forall$ " means "for all"