16 people are arranged in a circle and numbered consecutively from 1-16. Starting from #1, every 2nd person is eliminated.

- Elimination round #1: 2, 4, 6, 8, 10, 12, 14, 16
  All even numbers (16/2 = 8 numbers)

- Elimination round #2: 3, 7, 11, 15

- Elimination round #3: 5, 13

- Elimination last round: 9. \( \Rightarrow \) #1 is the only remaining person.

2) Mathematical induction:
Prove that for any \( n \in \mathbb{Z}^+ \), given \( 2^n \) people arranged in a circle and numbered from 1-\( 2^n \) consecutively, if every 2nd person starting from #1 is eliminated, only #1 remains.

**Basis step:** \( n = 1 \Rightarrow 2 \) persons are in the circle \( \Rightarrow \) obviously only #1 remains after elimination.

**I.H.:** Given \( 2^n \) people, only #1 remains.

**To show:** Given \( 2^{n+1} \) people, only #1 remains.

Take \( 2^{n+1} \) people arranged in the circle. After round 1, it is clear that all the "even" people are eliminated \( \Rightarrow \frac{2^{n+1}}{2} = 2^n \) even people are gone and \( 2^n \) "odd" people remain.
Hence we are left with \( 2^n \) people in the circle, but by induction hypothesis, then only \( \#1 \) remains after the elimination process.

3) Let \( r = 2^n + m \), \( 0 \leq m < 2^n \)

Claim: if the \( r \) people are arranged in a circle, only \( \#(2m+1) \) remains after the elimination process discussed previously.

Proof: Start by removing the \( m \) people numbered \( 2, 4, 6, \ldots, 2m \) in the first round. You are left then with \( r - m = 2^n \) people and your "new" \( \#1 \) is person \( 2m+1 \). By the previous result, only person \( 2m+1 \) remains.

Practice midterm:

# 4: Show that \( 13^n \equiv 6^n \pmod{7} \) \( \forall n \in \mathbb{Z}^+ \) (PMT)

Proof: Base step: \( n = 1 \):
\[
13^1 = 6^1 \pmod{7}
\]
\[
\iff \quad 7 \mid 13 - 6 = 7 \quad \checkmark
\]

\( n \rightarrow n+1 \):

\[13^n \equiv 6^n \pmod{7} \implies 7k = 13^n - 6^n \]

\[\text{for some } k \in \mathbb{Z}, \quad 7 \mid 13^n - 6^n \]

To show: \( 13^{n+1} \equiv 6^{n+1} \pmod{7} \)

\[7 \mid 13^{n+1} - 6^{n+1} \]

\[\iff \quad 7k = 13^{n+1} - 6^{n+1} \quad \text{for some } k \in \mathbb{Z} \]

But \( 13^{n+1} - 6^{n+1} = 13 \cdot 13^n - 6 \cdot 6^n = 13(7k + 6^n) - 6 \cdot 6^n \]

\[\implies \quad 7 \mid 13^{n+1} - 6^{n+1} \quad \checkmark \]

\[\implies \quad 7 \mid 13^{n+1} - 6^{n+1} \quad \iff \quad k \in \mathbb{Z} \]
Show that \( \forall n \in \mathbb{Z}^+ \), \( \exists a, b \in \mathbb{Z}^+ \), such that \( n = 2^{a-1}(2b-1) \) (SMI).

**Basis step**: \( n = 1 \). Let \( a = 1, b = 1 \)
\[
2^{a-1}(2b-1) = 2^0 \cdot 1 = 1 = n. \quad \checkmark
\]

**I.H.**: \( \forall k \in \mathbb{Z}^+, \underline{2 \leq k \leq n}, n \in \mathbb{Z}^+, \exists a, b \in \mathbb{Z}^+, \) such that \( n = 2^{a-1}(2b-1) \).

**n \rightarrow n+1**: to show: \( n+1 \) can also be written in that form.

**Case 1**: \( n+1 \) is even \( \Rightarrow n+1 = 2k \), for some \( k \in \mathbb{Z}^+ \).

By I.H., since \( k \leq n \), \( k \in \mathbb{Z}^+ \), we get:
\[
\frac{n+1}{2} = k = 2^{a-1}(2b-1) \text{ for some } \tilde{a}, \tilde{b} \in \mathbb{Z}^+.
\]
\[
\Rightarrow n+1 = 2k = 2^{\tilde{a}+1-1}(2\tilde{b}-1).
\]

**Case 2**: \( n+1 \) is odd \( \Rightarrow n+1 = 2k+1 \), for some \( k \in \mathbb{Z}^+ \).

Here no need of induction. Choose \( a = 1, b = k+1 \)
\[
\Rightarrow 2^{1-1}(2(2k+1)-1) = 2k+1 = n+1. \quad \checkmark
\]

Other problem involving Strong induction.

**Q**. Prove that for any integer \( n \geq 8 \), there are integers \( a, b \in \mathbb{Z} \), \( n \geq 8 \), such that \( n = 3a + 5b \).
\textbf{Proof:}

**Base step:** \( n = 8 \). Choose \( a = 1, b = 1 \): \( 3 \cdot 1 + 5 \cdot 1 = 8 \)

**I.H:** Assume \( k = 3a + 5b, a, b \in \mathbb{Z}^\neq0, \forall k \in \mathbb{Z}, 8 \leq k \leq n, n \in \mathbb{Z} \).

\( n \rightarrow n+1 \): To show: \( n+1 = 3a + 5b \) for some \( a, b \in \mathbb{Z}^\neq0 \).

Consider \((n+1)-3 = n-2\)

**Case 1:** \( n-2 < 8 \) since \( n \geq 8 \), this occurs only if \( n = 8 \) or \( n = 9 \).

\( P(8) \) was shown already (Base step).

\( P(9): 9 = 3 \cdot 3 + 5 \cdot 0 \) \( (a = 3, b = 0) \Rightarrow P(8+1) \) true

**Case 2:** \( n-2 \geq 8 \), then by I.H.:

\((n+1)-3 = n-2 = 3a + 5b \) for some \( a \) and \( b \).

\( \Rightarrow (n+1) = 3a + 3 + 5b \)

\[ = 3(a+1) + 5b \]

\[ = \alpha + 5b \]

\[ = \alpha + 5b \]

\( \therefore \)

Be careful with induction.

The base step is essential.

\[ \sum_{k=0}^{n} 2k + 3 = n^2 + 4n \] is wrong but the induction step works, although the statement is false for \( n = 1, 2, 3, \ldots \).