CHAPTER 0

Introduction

Mathematics has an advantage over other subjects. Theorems are absolute. They are not subject to further discussion as to their correctness. No sane person can write a paper disputing the Pythagorean theorem and expect to be taken seriously. Once you have proved a theorem, it is forever true. On the other hand your theorem will not be accepted until it is proved. You cannot argue that you have checked 79 right triangles and have always found $a^2 + b^2 = c^2$ and thus have proved the Pythagorean theorem. Your study of examples is good work and leads you to conjecture that $a^2 + b^2 = c^2$ in any right triangle. But the result will only be a mathematical conjecture until it is proved. Some conjectures in mathematics have been checked for trillions of cases and mathematicians are 99.9999% certain they are valid but until a proof is supplied they are considered unsolved problems.

A theorem is not established until it is proved. The statement in the theorem must be shown to always hold. A proof is a thoroughly convincing explanation, an explanation that leaves no shadow of doubt, as to why something is true. The amount of explanation and detail in this explanation depends upon the audience for the proof. The amount of explanation required between professors might be much less than that required between a professor and a student. Of course a purported theorem (a conjecture) can be shown to be false by the presentation of just one exception to the statement.

Let’s suppose that your friend Emile says that he has discovered the following two part theorem.

Emile’s Theorem. A) The sum of any even integer and any odd integer is an odd integer.

B) The product of any even integer and any odd integer is an odd integer.

Emile, like ALL OF US, is sometimes wrong so you decide the best way to check this out is to try to prove the theorem yourself. For A) you first try to understand the statement by considering some examples: $5 + 4 = 9, 11 + 20 = 31, 2 + 1 = 3$. To prove the statement you need to know exactly what it says. This means knowing the definitions. What is an even integer? You could say “an even integer is any of the numbers 2, 4, 6, 8, . . .” Is this
correct? What about 0? Is 0 even? What about $-2$ or $-10$? Are they even. Yes, they are. Reading the statement of A) carefully we do not see the words “positive even integer” $(2, 4, 6, 8, \ldots)$ or “nonnegative even integer” $(0, 2, 4, \ldots)$. Now, is this listing of all even integers the right description to use in our proof? We could make a similar listing of all odd integers $(\ldots, -3, -1, 1, 3, 5, \ldots)$ but then to establish the result we need to show that if we add any number on the first list to any number on the second we have a number on the second list. We cannot check this by hand or by computer (there are infinitely many cases). So this is not the best definition of even and odd to use. However when you check your notes you see the following definitions:

**Definition.** An integer $m$ is **odd** if there exists an integer $n$ so that $m = 2n + 1$. An integer $m$ is **even** if there exists an integer $n$ so that $m = 2n$.

Now let’s give your proof of A):

**Proof.** Let $m_1$ be an even integer and let $m_2$ be an odd integer. Then there exist integers $n_1$ and $n_2$ so that $m_1 = 2n_1$ and $m_2 = 2n_2 + 1$. So $m_1 + m_2 = 2n_1 + 2n_2 + 1 = 2(n_1 + n_2) + 1$. So if $n_3 = n_1 + n_2$ then $m_1 + m_2 = 2n_3 + 1$. $n_3$ is an integer and so $m_1 + m_2$ is odd by the definition of odd.  

Note that your proof required you to choose some notation. You could not let $m_1 = 2n$ and $m_2 = 2n + 1$ as one might be tempted to do from the definition. Indeed this would mean that $m_2 = m_1 + 1$ which would be a restrictive case of the theorem and your proof would not be in full generality.

Next you try to understand Theorem B). You first try is $2 \cdot 3 = 6$. You can stop now! You have shown that “Theorem B)” is false. The statement means that if you multiply any even integer by any odd integer you get an odd integer. But $2 \cdot 3 = 6$ shows this to be false. You investigate further: $4 \cdot 7 = 28$, $9 \cdot 10 = 90$ and you posit a revised

**Theorem B)$'$. The product of an even integer and an odd integer is an even integer.

0.1. Prove Theorem B)$'$.  

0.2. Formulate and prove an improvement of Theorem B)$'$.  

In this course you will be proving many theorems. While they will often be more difficult than the examples above, the methodology you should use will be similar. First you should know all the definitions and previous results. This may seem impossible but if you have faithfully worked out the problems you will find that you DO know them. Maybe you
need a peek now and then to refresh your memory but they are your results. You own them. They are friendly companions. The harder you work and the more problems you solve the easier the next problem will be. It may still be hard and challenging but you can do it. Read the theorem. Read it again. Make sure you understand precisely what it says. What is assumed. What must be proved. Try to construct examples that illustrate the theorem. How can you use the hypothesis to deduce the conclusion? When you have finished your proof, review it carefully. What was your argument? How did your use the hypothesis? Did you actually prove what had to be proved? You will likely often find mistakes, glitches, omissions.... Rewrite your argument! As first written it probably looks like your cat scratched on the page. You want to make it neat, not just in appearance but in mathematical organization. You will then likely want to rewrite it a third time. Take pride in your work. A well written explanation will serve you well as the course progresses. It makes for easy review and future use. It will be much much easier to present in class.

Now all of this takes practice and effort and time. As we go along you will become better and better at it, unless you don’t do the work. You cannot expect to succeed on a basketball team unless you attend practice, exercise and keep in shape and play the games. You should always come to class, do the homework in advance of class and come to class prepared to show your solutions. In class when others are presenting you should listen attentively to their arguments. If you don’t understand something, ask a question. If you think you see an error, politely point it out.

These notes are your pre-text. More will be distributed as needed. This pre-text along with your solutions to the problems will form your text. You are writing you own textbook!

Before we start with chapter 1, let’s discuss some general things from logic. Many of the theorems/problems you will be asked to prove/solve are of the form \((P) \Rightarrow (Q)\), or \((P)\) implies \((Q)\). \((P)\) is the hypothesis and \((Q)\) is the conclusion. The “theorem” \((P) \Rightarrow (Q)\) is false only if \((P)\) holds and \((Q)\) does not. The theorem \((P) \Leftrightarrow (Q)\) is actually two statements: \((P) \Rightarrow (Q)\) and \((Q) \Rightarrow (P)\) and often you will need to write separate proofs for each part. We read \((P) \Leftrightarrow (Q)\) as \((P)\) if and only if \((Q)\)”. \((P)\) only if \((Q)\)” means \((P) \Rightarrow (Q)\). \((P)\) if \((Q)\)” means \((Q) \Rightarrow (P)\). The converse of \((P) \Rightarrow (Q)\) is \((Q) \Rightarrow (P)\). The contrapositive of \((P) \Rightarrow (Q)\) is not \(Q \Rightarrow \neg (P)\). The contrapositive of an implication is logically equivalent to the implication: both are valid or both are false.

0.3. The theorem “If John is 6'5" tall then he is smart” is true if and only if “?”. Replace ? by the contrapositive.
Some statements will involve quantifiers. “∀” means “for all” and “∃” means “there exists”. Sometimes these quantifiers may be hidden in a statement. Consider the

**Theorem.** *If x is an odd integer then x² is an odd integer.*

This involves the universal quantifier, ∀. It can be restated as (Z denotes the set of all integers)

“∀ x ∈ Z, if x is odd then x² is an odd integer.”

(Here x ∈ Z just means x is an element of the set Z or x is an integer.)

**0.4.** Prove or disprove each of the following.

a) If x is an integer then x² > 9 if and only if x > 3.

b) If 5 > 3 then 4 is an even integer.

c) If 5 < 3 then 4 is an odd integer.

d) If 5 is an even integer then 4 > 3.