A set is a well defined collection of objects. What does this mean? Well there is an entire area of mathematics called Set Theory which is primarily concerned with deciding what this means. Shortly we shall consider what the the difficulty might be, but first we need some notation. If $A$ is a set, the “objects” in $A$ are called elements of $A$ and we write “$x \in A$” or “$x$ belongs to $A$” to denote that $x$ is an object or element in $A$. Sets can be denoted or described in various ways.

$A = \{1, 2, 3\}$ has 3 elements, namely 1, 2 and 3.

$\mathbb{N} = \{1, 2, 3, \ldots\}$ denotes the set of natural numbers. Note “$0 \notin \mathbb{N}$”, or “0 is not an element of $\mathbb{N}$” by definition of $\mathbb{N}$.

$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ denotes the set of integers.

$\mathbb{Q} = \{\frac{n}{m} : n \in \mathbb{Z}, m \in \mathbb{Z} \text{ and } m \neq 0\}$ denotes the set of rational numbers. The : is read “such that”. Thus $\mathbb{Q}$ is “the set of all $\frac{n}{m}$ such that $n$ and $m$ are integers with $m \neq 0$.”

$\mathbb{R}$ denotes the set of all real numbers. \(\{x \in \mathbb{R} : x^{13} - 27x^{12} + 16x^2 - 4 = 0\}\) is the set of all real roots of the given polynomial. This set is well defined even if we cannot say precisely what those elements are.

Let $A = \{\{1\}, 2, 3\}$. How many elements are in $A$? What are they?

So a set can be an element of another set. Consider the following and ask yourself if it is a set.

$A = \{1, \{\ldots\}\}$.

Well $A$ appears to have two elements, namely 1 and a second set of which we do not known anything so $A$ is not yet a well described collection of objects.

Let’s look closer.

$A = \{1, \{1, \{\ldots\}\}\}$.

Is $A$ now a set? Look even closer.

$A = \{1, \{1, \{1, \{1, \{\ldots, \}\}\}\}\}$.

The intent of this strange notation seems clear. We keep going forever with the brackets and the 1’s. Is $A$ a set? If so then $A$ has two elements. What are they? Well 1 and uh, well, I
guess $A$. So $A \in A$! Can this happen? Is $A$ a set? Is this a well defined collection of objects? Bertam Russell considered this problem and the following exercise is due to him. Let us say a set $A$ is normal if $A \notin A$. $A$ is abnormal if $A \in A$. Let $N = \{ A : A $ is a normal set $\}$. Is $N$ a set?

1.1. Show that if $N$ is normal then $N$ is abnormal. Show that if $N$ is abnormal then $N$ is normal. (This is called Russell’s paradox.)

At this point we will leave it to the set theorists to ponder such things. We will be working in this course with well defined sets that do not pose such difficulties.

**Basic set theory notation and definitions.**

i) $\emptyset$ is the empty set, i.e. the set with no elements.

ii) $A \subseteq B$ means that “$\forall x, x \in A \Rightarrow x \in B$”

It is read “$A$ is a subset of $B$”.

iii) $A = B$ means that $A$ and $B$ have the same elements.

iv) $A \cap B = \{ x : x \in A \text{ and } x \in B \}$

$A \cap B$ is read as “$A$ intersect $B$” and is called the intersection of $A$ and $B$.

v) $A \cup B = \{ x : x \in A \text{ or } x \in B \}$

$A \cup B$ is read as “$A$ union $B$” and is called the union of $A$ and $B$. If $x \in A \cup B$ then $x \in A$ or $x \in B$. It could be in both.

vi) $A \setminus B = \{ x : x \in A \text{ and } x \notin B \}$

$A \setminus B$ is read as “$A$ minus $B$” and is called the “set theoretic difference of $A$ and $B$”.

vii) $A \triangle B = (A \setminus B) \cup (B \setminus A)$ and is called the symmetric difference of $A$ and $B$.

viii) $C(A) = \{ x : x \notin A \}$ is the complement of $A$.

Caution: To be honest this is not really a set since we have not said what $x$ is other than it is not in $A$. We don’t want in this course for example oranges to be in $C(A)$. When we use $C(A)$ we will have a (usually unspecified) universe $U$ in mind so actually $C(A) = \{ x \in U : x \notin A \}$ and $A$ will be a subset of $U$. So $C(A) = U \setminus A$.

We can form the union of more than one set

ix) $\bigcup_{i=1}^{n} A_i = \{ x : \exists i \text{ with } 1 \leq i \leq n \text{ and } x \in A_i \}$

x) $A$ and $B$ are disjoint if $A \cap B = \emptyset$

1.2. Formulate the definitions for $\bigcap_{i=1}^{n} A_i$, $\bigcap_{i=1}^{\infty} A_i$ and $\bigcup_{i=1}^{\infty} A_i$.

1.3. Prove or disprove:
a) If $A$ is a set $\emptyset \subseteq A$ and $A \subseteq A$.

b) If $A$ and $B$ are sets then $A = B \iff A \subseteq B$ and $B \subseteq A$.

c) $\emptyset \in \emptyset$

1.4. Prove each of the following.

a) $C(A \cap B) = C(A) \cup C(B)$

b) $C(A \cup B) = C(A) \cap C(B)$

c) $A \triangle B = (A \cup B) \setminus (A \cap B)$

d) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

e) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

f) $C(C(A)) = A$

1.5. True or false? Your answer should include an explanation.

a) $\{1, 2, 1\} = \{2, 1\}$

b) $\{1, 1, 1, \ldots\} = \{1\}$

1.6. Using only sets define an ordered pair $(a, b)$. Note: you cannot say $(a, b) = \{a, b\}$.

Why?

xii) If $A$ and $B$ are sets the Cartesian product of $A$ and $B$ is

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$$

where $(a, b)$ denotes the ordered pair.

NOTATION. $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ is the Cartesian plane.

We can define unions and intersections over arbitrary sets.

xiii) If $I$ is a set and for all $i \in I$, $A_i$ is a set then $\bigcup_{i \in I} A_i = \{x : \exists i \in I, x \in A_i\}$ and $\bigcap_{i \in I} A_i = \{x : \forall i \in I, x \in A_i\}$.

1.7.

a) Describe in simpler terms

$$\bigcup_{r \in \mathbb{R}} \{(x, y) : x + y = r\}.$$  

Prove that if $I$ is a set and $\forall i \in I$, $A_i$ is a set then

b) $C(\bigcup_{i \in I} A_i) = \bigcap_{i \in I} C(A_i)$.

c) $C(\bigcap_{i \in I} A_i) = \bigcup_{i \in I} C(A_i)$.  

Functions. Roughly speaking a function $f$ from $A$ to $B$ is a rule that assigns to each $x \in A$ an element $f(x) \in B$. In this case we write $f : A \to B$. What do we mean by “rule”? Let’s try to be more precise.

Definition. A function $f$ from $A$ to $B$ is a subset $f \subseteq A \times B$ satisfying

i) $\forall a \in A \exists b \in B$ so that $(a, b) \in f$

ii) $\forall a \in A \forall b, b' \in B$, if $(a, b) \in f$ and $(a, b') \in f$ then $b = b'$.

Notation. If $(a, b) \in f$ we write $b = f(a)$.

Let $f : A \to B$. $A$ is called the domain of $f$. $B$ is called the co-domain. The range of $f$ is denoted by $f(A)$ and defined by

$$f(A) = \{ f(a) : a \in A \}$$

Most of the functions we consider will be of the form $f : \mathbb{R} \to \mathbb{R}$ or $f : A \to \mathbb{R}$ where $A \subseteq \mathbb{R}$.

1.8. In your past mathematical life you encountered “the vertical line test”. Explain what this is all about.

A function is sometimes called a mapping or a transformation. If $f(a) = b$ we might say “$f$ maps $a$ to $b$” or “$f$ sends $a$ to $b$”.

Definition. Let $f : A \to B$.

a) $f$ is onto (or surjective) if $f(A) = B$

b) $f$ is 1–1 (or one-to-one or injective) if $\forall a_1, a_2 \in A$, $f(a_1) = f(a_2)$ implies $a_1 = a_2$.

c) $f$ is a bijection if it is 1–1 and onto.

1.9. What is the “horizontal line test” and what does it have to do with 1–1 functions?

1.10. Let $f : A \to B$ be a bijection. Show that there exists a function $g : B \to A$ satisfying

a) $\forall a \in A$, $g(f(a)) = a$

b) $\forall b \in B$, $f(g(b)) = b$

c) If $h : B \to A$ satisfies (replacing $g$ by $h$) both a) and b) then $h = g$.

Terminology. The function $g$ produced in 1.10 is called the inverse function of $f$ and is usually denoted by $f^{-1}$.

Definition. Let $f : A \rightarrow B$. Let $C \subseteq A$ and $D \subseteq B$.

a) The direct image of $C$ under $f$ is $f(C) = \{ f(x) : x \in C \}$.

b) The inverse image of $D$ under $f$ is $f^{-1}(D) = \{ a \in A : f(a) \in D \}$.
c) The graph of \( f \) is \( G(f) = \{(a, f(a)) : a \in A\} \subseteq A \times B \).

1.11. Which of the following sets \( G \) are the graph of a function? If so identify the domain and range. Is the function 1–1 or onto?

a) \( G = \{(a, a) : a \in \mathbb{R} \text{ and } a > 0\} \)

b) \( G = \{(x^2, a) : a \in \mathbb{R}\} \)

c) \( G = \{(x, y) : x^2 + y^2 = 1 \text{ and } y \leq 0\} \)

d) \( G = \{(x, \sin x) : x \in \mathbb{R}\} \)

e) \( G = \{(x, \sin x) : -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}\} \)

Interval Notation. Let \( a \leq b \) be real numbers. Then

i) \( (a, b) = \{x \in \mathbb{R} : a < x < b\} \)

ii) \( (a, b] = \{x \in \mathbb{R} : a < x \leq b\} \)

iii) \( [a, b) = \{x \in \mathbb{R} : a \leq x < b\} \)

iv) \( [a, b] = \{x \in \mathbb{R} : a \leq x \leq b\} \)

v) \( [a, \infty) = \{x \in \mathbb{R} : a \leq x\} \)

vi) \( (a, \infty) = \{x \in \mathbb{R} : a < x\} \)

vii) \( (-\infty, b] = \{x \in \mathbb{R} : x \leq b\} \)

viii) \( (-\infty, b) = \{x \in \mathbb{R} : x < b\} \)

ix) \( (-\infty, \infty) = \mathbb{R} \)

Note. If \( a = b \) we can still define \([a, a] = \{a\} \). The intervals in i), vi), viii) and ix) are called open intervals. The intervals in iv), v) and vii) are called closed intervals.

1.12. Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be given by \( f(x) = x^2 \). Find

a) \( f([0, 2]) \)

b) \( f([-1, 2]) \)

c) \( f^{-1}((2, 3]) \)

d) \( f^{-1}([-1, 2]) \)

Definition. Let \( f : A \rightarrow B \) and \( g : B \rightarrow C \). The composition \( g \circ f : A \rightarrow C \) is defined by \((g \circ f)(a) = g(f(a))\).

Construct some examples to help yourself understand the definition.

1.13. Prove or disprove: If \( f : A \rightarrow A \) and \( g : A \rightarrow A \) then \( f \circ g = g \circ f \).

Definition. Let \( A \) be a set. The power set of \( A \) is defined as

\[ \mathcal{P}(A) = \{B : B \subseteq A\} \].

a) Show the direct image definition yields a function, also denoted by $f, f : \mathcal{P}(A) \to \mathcal{P}(B)$.

b) Suppose $f : A \to B$ is 1–1. Is the direct image function of part a) also 1–1?

c) Suppose $f : A \to B$ is onto. Is the direct image function of part a) also onto?

Caution. Let $f : A \to B$. Then the inverse image function $f^{-1} : \mathcal{P}(B) \to \mathcal{P}(A)$ always exists. The inverse function $f^{-1} : B \to A$ will exist only if $f$ is 1–1 onto. Since we use the same symbol for both, confusion can occur unless you are diligent.

The Theorem of Mathematical Induction. Let $P(1), P(2), P(3), \ldots$ be a list of statements, each of which is either true or false. Suppose that

i) $P(1)$ is true

ii) $\forall n \in \mathbb{N}, P(n) \Rightarrow P(n + 1)$.

Then $\forall n \in \mathbb{N}, P(n)$ is true.

1.15. Prove this theorem.

Hint: Suppose it were not true. Choose $n_0$ to be the smallest integer so that $P(n_0)$ is false.

1.16. Use mathematical induction to establish the following

a) $\forall n \in \mathbb{N}, 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$

b) $\forall n \in \mathbb{N}, 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

c) $\forall n \in \mathbb{N}$, if $n \geq 4$ then $2^n < n!$.

Note. $n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot (n-1) \cdot n$. This is called “$n$ factorial.”

Cardinality.

Definition. If $A$ and $B$ are sets we write $|A| = |B|$ if there exists a 1–1 onto function $f : A \to B$. We read $|A| = |B|$ as “the cardinality of $A$ equals the cardinality of $B$.

Intuitively this means that both sets have the “same number of elements”. This is not startling for finite sets. It is no surprise that $|\{a, b, c\}| = |\{1, 2, 3\}|$. However this definition can lead to non-intuitive results.

1.17. Prove that

a) $|\mathbb{N}| = |\{2, 4, 6, 8, \ldots\}|$

b) $|\mathbb{N}| = |\mathbb{Z}|$

c) $|\mathbb{N}| = |\{x \in \mathbb{Q} : x > 0\}|$
1. SETS AND FUNCTIONS

Hint: Try to make an infinite list of all rationals in (0, 1). Now try to make a list of all rationals > 1.

d) \(|\mathbb{N}| = |\mathbb{Q}|\).

**DEFINITION.** A is *finite* if \(A = \emptyset\) or if there exists \(n \in \mathbb{N}\) with \(|A| = |\{1, 2, \ldots, n\}|\). (We then say \(|A| = 0\) or \(|A| = n\) accordingly.) A is *infinite* if A is not finite. A is *countably infinite* if \(|A| = |\mathbb{N}|\). A is *countable* if A is finite or countably infinite.

1.18. Prove that a set \(A\) is

a) countably infinite if and only if we can write \(A = \{a_1, a_2, \ldots\}\) where \(a_i \neq a_j\) if \(i \neq j\).

b) countable if and only if \(A = \emptyset\) or we can write \(A = \{a_1, a_2, \ldots\}\).

1.19. If \(A\) is countable and \(B\) is countable prove that \(A \times B\) is countable.

Hint: You want to construct a list of all elements in \(A \times B\) (see 1.8). Can you make an infinite matrix of these elements starting with

\[
\begin{array}{ccc}
  a_1 & a_2 & a_3 & \ldots \\
  b_1 & a_2 & a_3 & \ldots \\
  b_2 & a_3 & \ldots \\
  b_3 & \vdots & & \\
  \vdots & & & \\
\end{array}
\]

Can you take this matrix and make a list as in 1.8 b)?

1.20. a) Prove that (0, 1) is not countable.

Hint: If it were countable then (0, 1) = \(\{a_1, a_2, a_3, \ldots\}\). Write each \(a_i\) as a decimal to get an infinite matrix as the following example illustrates.

\[
\begin{align*}
  a_1 &= 0.13974 \cdots \\
  a_2 &= 0.000002 \cdots \\
  a_3 &= 0.55556 \cdots \\
  a_4 &= 0.345587 \cdots \\
  a_5 &= 0.9871236 \cdots \\
  \vdots & & 
\end{align*}
\]

Can you find a decimal in (0, 1) that is not on this list? Can you describe an algorithm for producing such a number? Could \(a = 0.5 \cdots\) be equal to \(a_1\)? Could \(a = 0.54 \cdots\) be equal to \(a_1\) or \(a_2\)?

b) If \(a < b\) show that \(|(0, 1)| = |(a, b)| = |[0, 1]| = |[a, b]|\).
Definition. \( x \in \mathbb{R} \) is irrational if \( x \notin \mathbb{Q} \).

1.21. a) Prove that if \( A \) and \( B \) are countable then \( A \cup B \) is countable.
   
b) Prove that \( \mathbb{R} \setminus \mathbb{Q} \) is uncountable (i.e., not countable).
   
c) Prove that if \( a < b \) then \( (a, b) \cap (\mathbb{R} \setminus \mathbb{Q}) \neq \emptyset \).
   
d) Prove that if \( I \) is countable and \( \forall i \in I, A_i \) is a countable set then \( \bigcup_{i \in I} A_i \) is countable.

1.22. a) Let \( f : A \to B \) and \( g : B \to C \) be bijections. Prove that \( g \circ f : A \to C \) is a bijection.
   
b) Let \( |A| = |B| \) and \( |B| = |C| \). Prove that \( |A| = |C| \).