Solution to Homework 1

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Section 1.1: 4, 5(b)(d), 7(b)(f), 8(a)(b), 22; Section 1.2: 2, 5, 7, 9, 10, 11(b)(c), 12, 15(b), 23

Section 1.1

4. In each of the following cases, find a point that is two-thirds of the distance from the first (initial) point to the second (terminal) point.

(a) \((-4, 7, 2), (10, -10, 11)\)

Solution: Given two points \(P\) and \(Q\), the vector that represents the movement from \(P\) to \(Q\) is \(P - Q\). We are looking for a point \(A\) which is two thirds of the distance from \(P\) to \(Q\) - this is equivalent to saying that we start at \(P\), and go two-thirds of the way to \(Q\). Going all the way to \(Q\) corresponds to \(P - Q\) – this means that going to \(A\) corresponds to going \(\frac{2}{3}(P - Q)\).

In this particular case, we see that the vector corresponding to the motion from the first point to the second point is \((10, -10, 11) - (-4, 7, 2) = [14, -17, 9]\). This means that the vector corresponding to going two-thirds of the way is

\[ \frac{2}{3}[14, -17, 9] = [28/3, -34/3, 6] \]

Therefore, to get from \(P\) to \(A\) we need to move according to the above vector. Thus, \(A - P = [28/3, -34/3, 6]\), and so

\[ A = P + [28/3, -34/3, 6] = (-4, 7, 2) + [28/3, -34/3, 6] = [16/3, -13/3, 8] \]

(b) \((2, -1, 0, -7), (-11, -1, -9, 2)\).

Solution: Doing the same kind of calculation as in part (a), we see that our point is precisely:

\[ (2, -1, 0, -7) + \frac{2}{3} ((-11, -1, -9, 2) - (2, -1, 0, -7)) \]
which simplifies to \((-20/3, -1, -6, -1)\).

5. In each of the following cases, find a unit vector in the same direction as the given vector. Is the resulting (normalized) vector longer or shorter than the original? Why?

(b) \([4, 1, 0, -2]\)

\textbf{Solution:} In order to find a unit vector in the same direction as a given vector, just divide by the length of the vector. (This makes intuitive sense: if you had a vector of length 2, then to get a unit vector (a vector of length 1) in the same direction, you would divide by 2.) Furthermore, it’s clear that our unit vector is longer than the original vector if and only if the vector we started with had length less than 1. Let \(\vec{u}\) be our unit vector. Then,

\[\vec{u} = \frac{[4, 1, 0, -2]}{\sqrt{4^2 + 1^2 + 0^2 + (-2)^2}} = \left[\frac{4}{\sqrt{21}}, \frac{1}{\sqrt{21}}, 0, \frac{-2}{\sqrt{21}}\right]\]

Since the length of the original vector \(\sqrt{21}\), the resulting unit vector is shorter than the original vector.

(d) \([\frac{1}{5}, -\frac{2}{5}, -\frac{1}{5}, \frac{2}{5}]\)

\textbf{Solution:} In the same way as before,

\[\vec{u} = \frac{[\frac{1}{5}, -\frac{2}{5}, -\frac{1}{5}, \frac{2}{5}]}{\sqrt{(-\frac{2}{5})^2 + (-\frac{1}{5})^2 + (\frac{1}{5})^2 + (\frac{2}{5})^2}} = \left[\frac{1}{\sqrt{11}}, -\frac{2}{\sqrt{11}}, -\frac{1}{\sqrt{11}}, \frac{2}{\sqrt{11}}\right] = \frac{\left[1, -\frac{2}{\sqrt{11}}, -\frac{1}{\sqrt{11}}, \frac{2}{\sqrt{11}}\right]}{\sqrt{11}}\]

Since the length of the original vector is \(\sqrt{11}\), the normalized vector is longer than the original vector.

7. If \(\vec{x} = [-2, 4, 5], \vec{y} = [-1, 0, 3]\), and \(\vec{z} = [4, -1, 2]\), find the following

(b) \(-2\vec{y}\)

\textbf{Solution:} By definition,

\[-2\vec{y} = -2[-1, 0, 3] = [2, 0, -6]\]
(f) $2\vec{x} + 3\vec{y} - 4\vec{z}$

Solution: By definition,

$$2\vec{x} + 3\vec{y} - 4\vec{z} = 2[-2, 4, 5] + 3[-1, 0, 3] - 4[1, -1, 2]$$

$$= [-4, 8, 10] + [-3, 0, 9] - [16, -4, 8]$$

$$= [-23, 12, 11]$$

8. Given $\vec{x}$ and $\vec{y}$ as follows, calculate $\vec{x} + \vec{y}$, $\vec{x} - \vec{y}$, and $\vec{y} - \vec{x}$ and sketch $\vec{x}, \vec{y}, \vec{x} + \vec{y}, \vec{x} - \vec{y}$ in the same coordinate system.

(a) $\vec{x} = [-1, 5], \vec{y} = [2, -4]$

Solution:

$$\vec{x} + \vec{y} = [-1, 5] + [2, -4] = [1, 1]$$

$$\vec{x} - \vec{y} = [-1, 5] - [2, -4] = [-3, 9]$$

$$\vec{y} - \vec{x} = [2, -4] - [-1, 5] = [3, -9]$$

(b) $\vec{x} = [10, -2], \vec{y} = [-7, -3]$

Solution:

$$\vec{x} + \vec{y} = [10, -2] + [-7, -3] = [3, -5]$$

$$\vec{x} - \vec{y} = [10, -2] - [-7, -3] = [17, 1]$$

$$\vec{y} - \vec{x} = [-7, -3] - [10, -2] = [-17, -1]$$

22. (a) Prove that the length of each vector in $\mathbb{R}^n$ is nonnegative.

Proof: 
Assumptions: $\vec{x}$ is a vector in $\mathbb{R}^n$.

Need to show: $||\vec{x}|| \geq 0$.

Let $\vec{x} = [x_1, x_2, \ldots, x_n]$. Then, $||\vec{x}|| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$. Since squares are nonnegative,

$$x_1^2 + x_2^2 + \cdots + x_n^2 \geq 0$$

Since the square root of a nonnegative number is defined to be nonnegative, we see that $||\vec{x}|| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} \geq 0$, so we’re done.

(b) Prove that the only vector in $\mathbb{R}^n$ of length 0 is the zero vector.
Proof:

Assumptions: \( \| \vec{x} \| = 0 \)

Need to show: \( \vec{x} = \vec{0} \)

Let \( \vec{x} = [x_1, x_2, \ldots, x_n] \). Then, we have that

\[ 0 = \| \vec{x} \| = \sqrt{x_1^2 + \cdots + x_n^2} \]

\[ \Rightarrow \quad 0 = x_1^2 + \cdots + x_n^2 \]

Since each of the \( x_i^2 \) is non-negative, and their sum is 0, the only way this is possible is that each is 0. Thus, \( x_i^2 = 0 \) for each \( i \), and hence \( \vec{x} = \vec{0} \).

Section 1.2

2. Show that the points \( A_1(9, 19, 16), A_2(11, 12, 13), \) and \( A_3(14, 23, 10) \) are the vertices of a right triangle. (Hint: Construct vectors between the points and check for an orthogonal pair.)

Solution: The vectors that represent the sides of the triangle are \( A_1 - A_2 \), \( A_2 - A_3 \), and \( A_3 - A_1 \). (We had to choose between \( A_1 - A_2 \) and \( A_2 - A_1 \), etc., but this is not important.) Accordingly, they are:

\[
A_1 - A_2 = (9, 19, 16) - (11, 12, 13) = [-2, 7, 3] \\
A_2 - A_3 = (11, 12, 13) - (14, 23, 10) = [-3, -11, 3] \\
A_3 - A_1 = (14, 23, 10) - (9, 19, 16) = [5, 4, -6]
\]

Checking, the various pairs, we see that

\[
(A_1 - A_2) \cdot (A_2 - A_3) = [-2, 7, 3] \cdot [-3, -11, 3] = -62 \\
(A_2 - A_3) \cdot (A_3 - A_1) = [-3, -11, 3] \cdot [5, 4, -6] = -77 \\
(A_3 - A_1) \cdot (A_1 - A_2) = [5, 4, -6] \cdot [-2, 7, 3] = 0
\]

Thus, the last equation shows that \( (A_3 - A_1) \) is perpendicular to \( (A_1 - A_2) \), and therefore the triangle has a right angle at \( A_1 \).

5. Why isn’t it true that if \( \vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^n \), then \( \vec{x} \cdot (\vec{y} \cdot \vec{z}) = (\vec{x} \cdot \vec{y}) \cdot \vec{z} \)?

Solution:

Since \( \vec{y} \cdot \vec{z} \) is a scalar, the dot product \( \vec{x} \cdot (\vec{y} \cdot \vec{z}) \) is simply not defined – it’s impossible to calculate the dot product of a vector and a scalar. Therefore, neither left-hand side nor right-hand side are defined, so the statement is not true.

7. Does the Cancellation Law of algebra always hold for the dot product: that is, assuming that \( \vec{z} \neq 0 \), does \( \vec{x} \cdot \vec{z} = \vec{y} \cdot \vec{z} \) always imply that \( \vec{x} = \vec{y} \)?
Solution: This does not hold. The simplest example one can come up with is an example where \( \mathbf{z} \) is perpendicular to both \( \mathbf{y} \) and \( \mathbf{x} \) – in that case, both the dot products are 0. For example, let \( \mathbf{x} = [0, 1, 1], \mathbf{y} = [0, 1, 0] \) and let \( \mathbf{z} = [1, 0, 0] \). In that case,
\[
\mathbf{x} \cdot \mathbf{z} = 0 = \mathbf{y} \cdot \mathbf{z}
\]
but \( \mathbf{x} \neq \mathbf{y} \).

9. Prove that if \( (\mathbf{x} + \mathbf{y}) \cdot (\mathbf{x} - \mathbf{y}) = 0 \) then \( ||\mathbf{x}|| = ||\mathbf{y}|| \).

Proof:
Assumptions: \( (\mathbf{x} + \mathbf{y}) \cdot (\mathbf{x} - \mathbf{y}) = 0 \)
Need to show: \( ||\mathbf{x}|| = ||\mathbf{y}|| \)

As usual, we use the assumption to show the desired result. Here, expanding things out,
\[
0 = (\mathbf{x} + \mathbf{y}) \cdot (\mathbf{x} - \mathbf{y}) = \mathbf{x} \cdot \mathbf{x} - \mathbf{y} \cdot \mathbf{y} + \mathbf{x} \cdot \mathbf{y} - \mathbf{y} \cdot \mathbf{x} = \mathbf{x} \cdot \mathbf{x} - \mathbf{y} \cdot \mathbf{y} = ||\mathbf{x}||^2 - ||\mathbf{y}||^2
\]
using the fact that \( \mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x} \) and the identity for length in terms of the dot product. Rearranging, this yields \( ||\mathbf{x}||^2 = ||\mathbf{y}||^2 \). Since the lengths are both nonnegative, we can take the square roots of both sides to conclude that \( ||\mathbf{x}|| = ||\mathbf{y}|| \), as required.

10. Prove that \( \frac{1}{2} \left( ||\mathbf{x} + \mathbf{y}||^2 + ||\mathbf{x} - \mathbf{y}||^2 \right) = ||\mathbf{x}||^2 + ||\mathbf{y}||^2 \).

Proof:
Assumptions: No real assumptions.
Need to show: \( \frac{1}{2} \left( ||\mathbf{x} + \mathbf{y}||^2 + ||\mathbf{x} - \mathbf{y}||^2 \right) = ||\mathbf{x}||^2 + ||\mathbf{y}||^2 \)

Here, there are no assumptions, so we just need to check that the left-hand side and the right-hand side are equal in general. Since we have squares of lengths, we rewrite everything in terms of dot products. We start from the left-hand side and manipulate it:
\[
\frac{1}{2} \left( ||\mathbf{x} + \mathbf{y}||^2 + ||\mathbf{x} - \mathbf{y}||^2 \right) = \frac{1}{2} \left( (\mathbf{x} + \mathbf{y}) \cdot (\mathbf{x} + \mathbf{y}) + (\mathbf{x} - \mathbf{y}) \cdot (\mathbf{x} - \mathbf{y}) \right)
\]
\[
= \frac{1}{2} (\mathbf{x} \cdot \mathbf{x} + 2 \mathbf{x} \cdot \mathbf{y} + \mathbf{y} \cdot \mathbf{y} + \mathbf{x} \cdot \mathbf{x} - 2 \mathbf{x} \cdot \mathbf{y} + \mathbf{y} \cdot \mathbf{y})
\]
\[
= \frac{1}{2} (2 \mathbf{x} \cdot \mathbf{x} + 2 \mathbf{y} \cdot \mathbf{y})
\]
\[
= ||\mathbf{x}||^2 + ||\mathbf{y}||^2
\]
as required.
11. (b) Prove that if $\vec{x}, \vec{y}, \vec{z}$ are mutually orthogonal vectors in $\mathbb{R}^n$, then

$$
\|\vec{x} + \vec{y} + \vec{z}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2 + \|\vec{z}\|^2
$$

**Proof:**

Assumptions: $\vec{x}, \vec{y}$ and $\vec{z}$ are mutually orthogonal: that is, $\vec{x} \cdot \vec{y} = 0, \vec{x} \cdot \vec{z} = 0,$ and $\vec{y} \cdot \vec{z} = 0$

Need to show: $\|\vec{x} + \vec{y} + \vec{z}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2 + \|\vec{z}\|^2$

Again, rewriting everything in terms of dot products:

$$
\|\vec{x} + \vec{y} + \vec{z}\|^2 = (\vec{x} + \vec{y} + \vec{z}) \cdot (\vec{x} + \vec{y} + \vec{z})
$$

$$
= \vec{x} \cdot \vec{x} + \vec{y} \cdot \vec{y} + \vec{z} \cdot \vec{z} + 2\vec{x} \cdot \vec{y} + 2\vec{y} \cdot \vec{z} + 2\vec{x} \cdot \vec{z}
$$

$$
= \|\vec{x}\|^2 + \|\vec{y}\|^2 + \|\vec{z}\|^2
$$

using the assumption, and the identity for the length of a vector.

(c) Prove that $\vec{x} \cdot \vec{y} = \frac{1}{4} (\|\vec{x} + \vec{y}\|^2 - \|\vec{x} - \vec{y}\|^2)$.

**Proof:**

Assumptions: None.

Need to show: $\vec{x} \cdot \vec{y} = \frac{1}{4} (\|\vec{x} + \vec{y}\|^2 - \|\vec{x} - \vec{y}\|^2)$.

Using dot products once again, and starting from the right-hand side:

$$
\frac{1}{4} (\|\vec{x} + \vec{y}\|^2 - \|\vec{x} - \vec{y}\|^2) = \frac{1}{4} ((\vec{x} + \vec{y}) \cdot (\vec{x} + \vec{y}) - (\vec{x} - \vec{y}) \cdot (\vec{x} - \vec{y}))
$$

$$
= \frac{1}{4} (\vec{x} \cdot \vec{x} + 2\vec{x} \cdot \vec{y} + \vec{y} \cdot \vec{y} - (\vec{x} \cdot \vec{x} - 2\vec{x} \cdot \vec{y} + \vec{y} \cdot \vec{y}))
$$

$$
= \frac{1}{4} (4\vec{x} \cdot \vec{y}) = \vec{x} \cdot \vec{y}
$$

as required.

12. Given $\vec{x}, \vec{y}, \vec{z}$ in $\mathbb{R}^n$, with $\vec{x}$ orthogonal to both $\vec{y}$ and $\vec{z}$, prove that $\vec{x}$ is orthogonal to $c_1\vec{y} + c_2\vec{z}$ where $c_1, c_2 \in \mathbb{R}$.

**Proof:**

Assumptions: $\vec{x}$ orthogonal to both $\vec{y}$ and $\vec{z}$; that is, $\vec{x} \cdot \vec{y} = 0$ and $\vec{x} \cdot \vec{z} = 0$.

Need to show: $\vec{x} \cdot (c_1\vec{y} + c_2\vec{z}) = 0$

The trickiest part here is writing everything down in terms of dot products! As soon as that’s done, it’s very easy. Using vector identities.

$$
\vec{x} \cdot (c_1\vec{y} + c_2\vec{z}) = \vec{x} \cdot (c_1\vec{y}) + \vec{x} \cdot (c_2\vec{z})
$$

$$
= c_1\vec{x} \cdot \vec{y} + c_2\vec{x} \cdot \vec{z} = 0
$$

and so we’re done.
15. Calculate $\text{proj}_a \vec{b}$ in each case, and verify that $\vec{b} - \text{proj}_a \vec{b}$ is orthogonal to $\vec{a}$.

(b) $\vec{a} = [-5, 3, 0], \vec{b} = [3, -7, 1]$.

Solution: By definition,

$$\text{proj}_a \vec{b} = \left( \frac{\vec{a} \cdot \vec{b}}{||\vec{a}||^2} \right) \vec{a}$$

Therefore, here we have

$$\text{proj}_a \vec{b} = \left( \frac{[-5, 3, 0] \cdot [3, -7, 1]}{||[-5, 3, 0]||^2} \right) [-5, 3, 0]$$

$$= \left( \frac{-36}{(-5)^2 + 3^2 + 0^2} \right) [-5, 3, 0]$$

$$= \frac{-36}{34} [-5, 3, 0] = \left[ \frac{90}{17}, \frac{-54}{17}, 0 \right]$$

Now, verifying that $\vec{b} - \text{proj}_a \vec{b}$ is orthogonal to $\vec{a}$:

$$\vec{a} \cdot (\vec{b} - \text{proj}_a \vec{b}) = [-5, 3, 0] \cdot \left( [3, -7, 1] - \left[ \frac{90}{17}, \frac{-54}{17}, 0 \right] \right)$$

$$= [-5, 3, 0] \cdot \left[ \frac{-39}{17}, \frac{65}{17}, 1 \right]$$

$$= \frac{195}{17} - \frac{195}{17} = 0$$

Since the dot product is 0, they are indeed orthogonal.

23. True or False:

(a) For any vectors $\vec{x}$ and $\vec{y}$, and any scalar $d$, $\vec{x} \cdot (d\vec{y}) = (d\vec{x}) \cdot \vec{y}$.

TRUE: This follows from Identity (4) in Theorem 1.5.

(b) For all $\vec{x}, \vec{y}$ in $\mathbb{R}^n$ with $\vec{x} \neq \vec{0}$, $\vec{x} \cdot \vec{y}/||\vec{x}|| \leq ||\vec{y}||$.

TRUE: This follows from Cauchy-Schwarz.

(c) For all $\vec{x}, \vec{y}$ in $\mathbb{R}^n$, $||\vec{x} - \vec{y}|| \leq ||\vec{x}|| - ||\vec{y}||$.

FALSE: In fact, the opposite of this is true, which can be shown using Triangle Inequality: $||\vec{x} - \vec{y}|| \geq ||\vec{x}|| - ||\vec{y}||$. Let’s provide a counterexample: let $\vec{x} = [1, 0]$ and $\vec{y} = [0, 1]$. In that case,

$$||\vec{x} - \vec{y}|| = ||[1, 1]|| = \sqrt{2}, ||\vec{x}|| = 1, ||\vec{y}|| = 1$$
and it’s not true that $\sqrt{2} \leq 1 - 1 = 0$.

(d) If $\theta$ is the angle between $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^n$, and $\theta > \frac{\pi}{2}$, then $\vec{x} \cdot \vec{y} > 0$.

**FALSE:** From Theorem 1.8, $\vec{x} \cdot \vec{y} > 0$ if and only if the angle $\theta$ is acute.

(e) The standard unit vectors in $\mathbb{R}^n$ are mutually orthogonal.

**TRUE:** The standard unit vectors are $\vec{e}_1 = [1, 0, \ldots, 0], \vec{e}_2 = [0, 1, 0, \ldots]$, etc., and it’s easy to check that any pair of these has dot product 0.

(f) If $\text{proj}_{\vec{b}}\vec{a} = \vec{b}$, then $\vec{a}$ is perpendicular to $\vec{b}$.

**FALSE:** It can be checked that $\vec{a}$ is perpendicular to $\vec{b}$ if and only if $\text{proj}_{\vec{b}}\vec{a} = \vec{0}$. Furthermore, $\text{proj}_{\vec{b}}\vec{b} = \vec{b}$ if and only if $\vec{a}$ and $\vec{b}$ are parallel. We will not prove these statements here, but we will provide a counterexample to the assertion. Take $\vec{a} = \vec{b} = [1, 0]$. In that case, it can be checked that $\text{proj}_{\vec{b}}\vec{b} = \vec{b}$, but $\vec{a}$ and $\vec{b}$ are certainly not orthogonal.