1. Check the 10 properties of vector spaces to see whether the following sets with the operations given are vector spaces. If they are vector spaces, give an argument for each property showing that it works; if not, provide an example (with numbers) showing a property that does not work.

(a) $V$ is the set of $2 \times 2$ matrices of the form

$$A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$$

with the operations defined as

$$A \oplus B = AB \text{ (normal matrix multiplication)}$$

$$c \odot A = cA$$

**Note:** Don’t forget that these definitions only apply to the $2 \times 2$ matrices in the form I wrote above!

(b) $V$ is $\mathbb{R}^2$, with addition and scalar multiplication defined as follows:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \oplus \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}$$

$$c \odot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} cx_1 \\ cx_2 \end{bmatrix}$$

(c) $V$ is the set of functions from $\mathbb{R}$ to the positive real numbers: that is, $V$ is the set of functions $f$ with domain $\mathbb{R}$ such that $f(x) > 0$ for all $x \in \mathbb{R}$. Let $f$ and $g$ be in $V$. We define $f \oplus g$ and $c \odot f$ by defining the values these functions take on at any $x \in \mathbb{R}$:

$$(f \oplus g)(x) = f(x)g(x)$$

$$(c \odot f)(x) = f(x)^c$$

**Note:** In questions like this, it’s good to be strict about your notation! Here, $f$ and $g$ are functions, while $f(x)$ and $g(x)$ are values that those functions take at $x$. So for example, notation like $f(x) \oplus g(x)$ doesn’t make any sense, because the operations $\oplus$ and $\odot$ are only defined on the elements in the vector space, and not on the values the functions take.
2. Check whether the following sets $W$ are non-empty subsets of $V$, closed under vector addition, and closed under scalar multiplication to check whether they are subspaces of the provided vector spaces $V$. If they are, show that everything works; if they are not, give an example that doesn’t work. (You do NOT need to check whether $V$ is a vector space!)

**Note:** If $V = \mathbb{R}^n$, we assume that it has the standard vector addition and scalar multiplication, and similarly for $V = M_{mn}$ (the set of $m \times n$ matrices.) Feel free to use the normal plus and scalar multiplication notation in those cases, instead of $\oplus$ and $\odot$.

(a) Is $W = \{c_1[1,2] + c_2[1,-1] \mid c_1, c_2 \in \mathbb{R}\}$ a subspace of $\mathbb{R}^2$? (Here, we’re allowing vectors in $\mathbb{R}^2$ to be row vectors.)

(b) Is $W = \{\text{the set of } 2 \times 2 \text{ diagonal matrices}\}$ a subspace of $M_{22}$?

(c) Let $W$ be the set of functions from $[0,1]$ to $\mathbb{R}$ such that $f(0) = 1$, and let $V$ be the vector space of all functions from $[0,1]$ to $\mathbb{R}$ with the vector addition and scalar multiplication defined normally, that is:

$(f \odot g)(x) = f(x) + g(x)$

$(c \odot f)(x) = cf(x)$

Is $W$ a subspace of $V$?

(d) Let $A$ be an $m \times n$ matrix, and let $\vec{b}$ be a vector in $\mathbb{R}^m$. For which values of $\vec{b}$ is $\{\vec{x} \mid A\vec{x} = \vec{b}\}$ a subspace of $\mathbb{R}^n$?

(e) Is $W = \{A \in M_{22} \mid |A| = 1\}$ a subspace of $M_{22}$?

3. Prove the following statements using the properties of vector spaces – at each step, name which property you’re using to justify it. Be careful not to use ‘intuitively obvious’ statements which aren’t in the properties!

(Yes, these are the statements that you were asked to show in class – I’m not sure most people got there, so I’m giving them again.)

(a) $a\vec{0} = \vec{0}$ for all scalars $a$

(b) $0\vec{v} = \vec{0}$ for all $\vec{v} \in V$.

(c) $(-1)\vec{v} = -\vec{v}$ for all $\vec{v} \in V$. 

2