1. Let $A$ and $B$ be defined as follows:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 3 & 5 \\ 1 & 6 & 11 \end{bmatrix}$$

(a) Demonstrate that $A$ and $B$ row equivalent by providing a sequence of row operations leading from $A$ to $B$.

(b) Check whether $\vec{x} = [1, 2, 3]$ is in the row space of $A$, and if it is, write it as a linear combination of the rows of $A$.

(c) Is $\vec{x}$ in the row space of $B$? (You shouldn’t need many calculations here...)

2. Prove that $R(AB) = R(A)B$ if $R$ is the row operation Row 1 $\rightarrow 2 \times$ Row 1.

**Hint:** Show that the $(i, j)$ entry of $R(AB)$ is equal to the $(i, j)$ entry of $R(A)B$. You’ll have to consider $i = 1$ and $i \neq 1$ separately!

3. Let $A$ be defined as follows:

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 5 & 3 & 11 \\ -2 & 1 & 0 \end{bmatrix}$$

In that case (you do not need to check this!), the row reduced echelon form of $A$ is

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

(a) What is the rank of $A$?

(b) Is $A$ singular or nonsingular?

(c) Check that if $\vec{b} = [2, 3, 1]^T$, then $\vec{x} = [0, 1, 0]^T$ solves the system $A\vec{x} = \vec{b}$.

(d) Using the information from parts (a), (b), and (c), without doing any calculations, how many solutions does $A\vec{x} = \vec{b}$ have? (Here, $\vec{b}$ is defined as in part (c).)
4. Prove that if $A$ is an $n \times n$ diagonal matrix whose row-reduced echelon form is $I_n$, then none of the diagonal entries of $A$ are 0.

5. Let $A$ be defined as below:

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 1 & 2 \\ 2 & 3 & 4 \end{bmatrix}$$

(a) Calculate $A^{-1}$ if $A$ is nonsingular, or prove that it is singular.
(b) Calculate $|A|$ by using row or column expansion.
(c) Calculate $|A|$ using row reduction (feel free to reuse your work from part (a) for this!)

6. Let $A$ and $B$ satisfy the following:


That is, we know some entries of $A$ and $B$ but not others.

(a) Prove that $A$ and $B$ are not inverses of each other.
(b) Show that $\vec{x} = [1, 2, 1]$ is not in the set $\{ \vec{x} \mid \vec{x} A = c[0, 1, 1], c \in \mathbb{R} \}$.

Note: Pay attention to the order of multiplication in the definition of that set!!

7. Let $A$ be the matrix defined as

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & -1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

(a) What is the characteristic polynomial $p_A(x)$ of $A$?
(b) What are the eigenvalues of $A$?
(c) Pick an eigenvalue of $A$, and write down the fundamental eigenvectors for that eigenvalue.

8. Prove that if

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

then

$$A^n = \begin{bmatrix} 1 & 2n \\ 0 & 1 \end{bmatrix}$$

for all positive integers $n$. 