Solutions to Homework 7

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1. Find the inverses of the following matrices:

(a) \[
\begin{bmatrix}
2 & 0 \\
0 & 3
\end{bmatrix}
\]

Solution:

Recall that we have a formula for inverses of 2 × 2 matrices, which is

\[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix}
d & -b \\
-c & a
\end{bmatrix}
\]

Therefore,

\[
\begin{bmatrix}
2 & 0 \\
0 & 3
\end{bmatrix}^{-1} = \frac{1}{6} \begin{bmatrix}
3 & 0 \\
0 & 2
\end{bmatrix} = \begin{bmatrix}
1/2 & 0 \\
0 & 1/3
\end{bmatrix}
\]

(b) \[
\begin{bmatrix}
a_{11} & 0 \\
0 & a_{22}
\end{bmatrix}
\]

if \(a_{11}, a_{22}\) are both non-zero.

Solution:

Just like above,

\[
\begin{bmatrix}
a_{11} & 0 \\
0 & a_{22}
\end{bmatrix}^{-1} = \frac{1}{a_{11}a_{22}} \begin{bmatrix}
a_{22} & 0 \\
0 & a_{11}
\end{bmatrix} = \begin{bmatrix}
1/a_{11} & 0 \\
0 & 1/a_{22}
\end{bmatrix}
\]

(c) \[
\begin{bmatrix}
-1 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & -2
\end{bmatrix}
\]

Solution:

Since the matrix is now 3 × 3, we need to augment with \(I_3\) and
row-reduce. Accordingly,

\[
\begin{bmatrix}
-1 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & -2
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

R_1 \rightarrow (-1) \times R_1

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

R_2 \rightarrow (1/2) \times R_1

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

R_1 \rightarrow (-1/2) \times R_1

Therefore, since the left-hand side row-reduced to the identity, we have that

\[
\begin{bmatrix}
-1 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & -2
\end{bmatrix}
- (-1)
= \begin{bmatrix}
1 & 0 & 0 \\
0 & 1/2 & 0 \\
0 & 0 & -1/2
\end{bmatrix}
\]

Therefore, since the left-hand side row-reduced to the identity, we have that

\[
\begin{bmatrix}
a_{11} & 0 & 0 & \cdots & 0 \\
0 & a_{22} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & a_{nn}
\end{bmatrix}
\)

\[\text{if } a_{11}, a_{22}, \ldots, a_{nn} \text{ are all non-zero.}

\textbf{Solution:}

Here, we would again need to augment the matrix by the identity matrix, and row-reduce the left-hand side. Now, to row-reduce a diagonal matrix to the identity, we would just need to divide Row \( i \) by \( a_{ii} \) for each \( i \). Since the same operation would be performed on the identity matrix, we would get that

\[
\begin{bmatrix}
a_{11} & 0 & 0 & \cdots & 0 \\
0 & a_{22} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & a_{nn}
\end{bmatrix}
- (-1)
= \begin{bmatrix}
1/a_{11} & 0 & 0 & \cdots & 0 \\
0 & 1/a_{22} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1/a_{nn}
\end{bmatrix}
\]

2. Show that if \( Ax = \vec{0} \) has a non-trivial solution, then \( A \) is singular.

\textbf{Proof:}

Let’s use the contrapositive! The contrapositive of “\( C \) implies \( D \)” is “not \( D \) implies not \( C \)” Since the question asks us to show that “\( Ax = \vec{0} \) implies that \( A \) is singular,” the contrapositive is “\( A \) is nonsingular implies that \( Ax = \vec{0} \) only has the trivial solution.” (Think about this until it makes perfect sense.)
Assume: $A$ is nonsingular.

Need to show: $A\vec{x} = \vec{0}$ only has the trivial solution.

Since $A$ is nonsingular, $A$ has an inverse $A^{-1}$. Now, consider the equation $A\vec{x} = \vec{0}$. We can certainly multiply both sides of this equation by $A^{-1}$. Therefore, the equation becomes

$$A^{-1}(A\vec{x}) = A^{-1}\vec{0}$$

Now, since matrix multiplication is associative, we have that $A^{-1}(A\vec{x}) = I_n\vec{x} = \vec{x}$. Since multiplying by the zero vector always yields the zero vector, we have that $A^{-1}\vec{0} = \vec{0}$. Therefore, we get that

$$\vec{x} = \vec{0}$$

and therefore the only solution to $A\vec{x} = \vec{0}$ is the trivial solution, as required.

3. (a) Give an example to show that $A + B$ can be singular if $A$ and $B$ are both nonsingular.

Solution:

This just requires some tinkering. A really easy solution is to take $A$ and $-A$ for any nonsingular matrix $A$. Then their sum is the zero matrix, which is patently singular. Therefore, define

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

It’s easy to check by row reduction that $A$ and $B$ are both row equivalent to the identity, and therefore are nonsingular. However,

$$A + B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

which is clearly singular.

(b) Give an example to show that $A + B$ can be nonsingular if $A$ and $B$ are both singular.

Solution:

Again, some easy tinkering yields an answer. Any matrix with a 0 row is singular. Therefore, take $A$ with only zeros in the first row, and $B$ with only zeros in the second row. For example, let

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$
Then, we have that

\[ A + B = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \]

which can be checked to be nonsingular, as required.

**Hint:** $2 \times 2$ matrices should suffice for this one!

4. For the matrix $A$ below, and for the values of $(i, j)$ provided, calculate the $(i, j)$ submatrix $A_{ij}$ and the $(i, j)$ minor $|A_{ij}|$.

$$ A = \begin{bmatrix} 1 & 1 & 3 \\ 4 & 7 & -1 \\ -1 & 10 & 3 \end{bmatrix} $$

(a) $(i, j) = (1, 1)$

**Solution:**

Crossing out the first row and first column, we get

\[ A_{11} = \begin{bmatrix} 7 & -1 \\ 10 & 3 \end{bmatrix} \]

Using the formula for the determinant of a $2 \times 2$ matrix, we get that

\[ |A_{11}| = 7 \cdot 3 - 10 \cdot (-1) = 17 \]

(b) $(i, j) = (2, 3)$

**Solution:**

Crossing out the second row and third column, we get

\[ A_{23} = \begin{bmatrix} 1 & 1 \\ -1 & 10 \end{bmatrix} \]

Using the formula for the determinant of a $2 \times 2$ matrix, we get that

\[ |A_{23}| = 1 \cdot 10 - (-1) \cdot 1 = 11 \]

(c) $(i, j) = (3, 1)$

**Solution:**

Crossing out the third row and first column, we get

\[ A_{31} = \begin{bmatrix} 1 & 3 \\ 7 & -1 \end{bmatrix} \]
Using the formula for the determinant of a $2 \times 2$ matrix, we get that
\[ |A_{31}| = 1 \cdot (-1) - 3 \cdot 7 = -22 \]

5. Calculate the determinants of the following matrices, using the methods learned in class:

(a) \[
\begin{bmatrix}
1 & 3 \\
-3 & 2
\end{bmatrix}
\]

Solution:

Using the formula for $2 \times 2$ determinants,
\[
\begin{vmatrix}
1 & 3 \\
-3 & 2
\end{vmatrix} = 1 \cdot 2 - 3 \cdot (-3) = 11
\]

(b) \[
\begin{bmatrix}
0 & 1 \\
-1 & 4 \\
2 & 5 \\
3 & -1
\end{bmatrix}
\]

Solution:

Expanding out along the first row,
\[
\begin{vmatrix}
0 & 1 & -1 \\
2 & 4 & 5 \\
3 & -1 & 1
\end{vmatrix} = 0 \cdot \begin{vmatrix} 4 & 5 & -1 \end{vmatrix} - 1 \cdot \begin{vmatrix} 2 & 5 \end{vmatrix} - 1 \cdot \begin{vmatrix} 3 & 1 \end{vmatrix} - 1 \cdot \begin{vmatrix} 2 & 4 \end{vmatrix} = -13 + (-14) = -27
\]

(c) \[
\begin{bmatrix}
1 & 0 \\
-1 & 3 \\
0 & 1 \\
1 & 1 & 1
\end{bmatrix}
\]

Solution:

Expanding out along the first row and skipping the 0s to save space:
\[
\begin{vmatrix}
1 & 0 & -2 & 0 \\
-1 & 3 & 0 & 1 \\
0 & 1 & 2 & 0 \\
1 & 1 & 1 & 1
\end{vmatrix} = 1 \cdot \begin{vmatrix} 3 & 0 & 1 \\
1 & 2 & 0 \\
1 & 1 & 1
\end{vmatrix} - 2 \cdot \begin{vmatrix} 0 & 1 & 1 \\
1 & 2 & 0 \\
1 & 1 & 1
\end{vmatrix} = -1 + 6 = 5
\]
Now, doing the second one by expanding out along the second row,

\[
\begin{vmatrix}
-1 & 3 & 1 \\
0 & 1 & 0 \\
1 & 1 & 1
\end{vmatrix} = 1 \cdot \begin{vmatrix}
-1 & 1 \\
1 & 1
\end{vmatrix} = -2
\]

Finally, plugging those back in,

\[
\begin{vmatrix}
1 & 0 & -2 & 0 \\
-1 & 3 & 0 & 1 \\
0 & 1 & 2 & 0 \\
1 & 1 & 1 & 1
\end{vmatrix} = 1 \cdot 5 + (-2) \cdot (-2) = 9
\]

6. Show that if \( A \) has a row that’s all 0s, then \( |A| = 0 \).

**Proof:**

*Assume:* \( A \) has a row that’s all 0s.

*Need to show:* \( |A| = 0 \).

Assume that Row \( i \) is all 0s (we know that some row is all 0s, but we don’t know which row – so we just give it a name). Then, expanding out the determinant along Row \( i \):

\[
|A| = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}
\]

Now, since Row \( i \) is all 0s, all the entries of this row are 0. Therefore, \( a_{i1} = 0, a_{i2} = 0, \ldots, a_{in} = 0 \). Thus,

\[
|A| = 0 \cdot A_{i1} + 0 \cdot A_{i2} + \cdots + 0 \cdot A_{in} = 0
\]

as required.

7. Prove that

\[
\begin{vmatrix}
1 & 1 & 1 \\
a & b & c \\
a^2 & b^2 & c^2
\end{vmatrix} = (a - b)(b - c)(c - a)
\]

(This kind of matrix is called a **Vandermonde matrix**.)

**Solution:**

The easiest thing to do here is to expand out both sides and check that they are equal. Working out the left-hand side first:

\[
\begin{vmatrix}
a & b & c \\
a^2 & b^2 & c^2
\end{vmatrix} = 1 \cdot \begin{vmatrix}
b & c \\
a^2 & c^2
\end{vmatrix} - 1 \cdot \begin{vmatrix}
a & c \\
a^2 & c^2
\end{vmatrix} + 1 \cdot \begin{vmatrix}
a & b \\
a^2 & b^2
\end{vmatrix}
\]

\[
= bc^2 - b^2c + a^2c - ac^2 + ab^2 - a^2b
\]
Now, for the right-hand side:

\[(a - b)(b - c)(c - a) = (ab - b^2 - ac + bc)(c - a)\]
\[= abc - b^2c - ac^2 + bc^2 - a^2b + ab^2 + a^2c - abc\]
\[= -b^2c - ac^2 + bc^2 - a^2b + ab^2 + a^2c\]

Comparing terms, we see that this is exactly the same as the left-hand side, so equality holds.

8. Show that if \(|B| < 0\), then \(B\) can’t be written as \(A^2\) for any matrix \(A\).

**Proof:**
We’re going to use the contrapositive! The contrapositive of “\(C\) implies \(D\)” is “not \(D\) implies not \(C\).” Our original statement translates to: “\(|B| < 0\) implies \(B\) can’t be written as \(A^2\) for any \(A\).” Therefore, the contrapositive is “\(B = A^2\) implies that \(|B| ≥ 0\).” (Think about this if you don’t get it!)

**Assume:** \(B = A^2\).
**Need to show:** \(|B| ≥ 0\).

For this proof, we will be assuming that \(A\) and \(B\) are square matrices, since otherwise \(B = A^2\) is impossible. Therefore, we can take the determinant of both sides, and use the fact that \(|CD| = |C||D|\) for any square matrices \(C\) and \(D\). (This is a theorem from the book and from lecture.)

\[|B| = |A^2| = |A \cdot A| = |A| \cdot |A| = |A|^2\]

Now, \(|A|\) is just some number that’s the determinant of \(A\). Since any number squared is nonnegative, we have that \(|A|^2 ≥ 0\). This shows that \(|B| ≥ 0\), so we’re done!

9. For the sets specified below, do the following:
- Give an example of an element of the set
- Check whether the element provided is in the set

(a) \(S = \{ \vec{x} \mid \vec{x} \cdot [1, 1, 0] = 0 \}\). Element to check: \(\vec{v} = [1, 1, 1]\).

**Solution:**
Let \(\vec{x} = [x_1, x_2, x_3]\). Then \(\vec{x}\) is in the set if

\[0 = \vec{x} \cdot [1, 1, 0] = [x_1, x_2, x_3] \cdot [1, 1, 0] = x_1 + x_2\]

Therefore, an example of a vector in this set is \([−1, 1, 0]\).

Checking, we have that \(\vec{v} \cdot [1, 1, 0] = [1, 1, 1] \cdot [1, 1, 0] = 2 ≠ 2\), so \(\vec{v}\) is not in the set.
(b) \( S = \{ A \mid A \text{ is a matrix of rank 1} \} \). Element to check: \( A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \).

**Solution:**

The rank of a matrix is the number of nonzero rows in its row-reduced echelon form. Therefore, an easy way to come up with an example is just to write \( A \) in this form. Thus, an example is

\[
A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}
\]

The row-reduced echelon form of the matrix provided is

\[
\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}
\]

This clearly has 2 nonzero rows, so the rank is 2. Therefore, this matrix is not in the set.

(c) \( S = \{ c_1[1, 1, 1] + c_2[0, 1, 1] \mid c_1, c_2 \in \mathbb{R} \} \). Element to check: \( \vec{v} = [1, 2, 1] \).

**Solution:**

To pick an example, just pick \( c_1 \) and \( c_2 \). We can let \( c_1 = c_2 = 1 \), in which case \( [1, 1, 1] + [0, 1, 1] = [1, 2, 2] \) is an example of an element of the set.

To check whether \( [1, 2, 1] \) is an element, we need to see if there exist \( c_1 \) and \( c_2 \) such that

\[
c_1[1, 1, 1] + c_2[0, 1, 1] = [1, 2, 1] \\
[c_1, c_1 + c_2, c_1 + c_2] = [1, 2, 1]
\]

This becomes the system of equations

\[
c_1 = 1 \\
c_1 + c_2 = 2 \\
c_2 + c_2 = 1
\]

which can be easily checked (or can even be seen) to have no solutions.

Thus, \( [1, 2, 1] \) is not in the set.