9.1 Watts-Strogatz small-world model

**Definition** A routing scheme from a source \( u \in V \) to a target \( v \in V \) is a path \( \{u(0),\ldots,u(k)\} \), where \( u(0) = u \), \( u(k) = v \), and \( u(j) \in N(u(j-1)) \forall j \leq k \).

**Definition** We say a routing is decentralized if \( u(k) \) is determined only with knowledge of \( \{u(0),\ldots,u(k)\} \cup \{N(u(0)),\ldots,N(u(k-1))\} \cup \{v\} \).

Our underlying model is a lattice (strong ties) with randomly added weak ties. We define the distance between two vertices of the lattice \( u = (u_1,v_1) \) and \( v = (v_1,v_2) \) to be

\[
d(u, v) = |u_1 - v_1| + |u_2 - v_2|
\]

**Definition** A greedy routing scheme with target \( v \) is one in which the \( k \)th node \( u(k) \), is chosen such that

\[
d(u(k), v) = \min\{d(u, v), u \in N(u(k-1))\}
\]

In other words, at each node, we choose the neighbor that gets us closest to the target in terms of taxi-cab distance.

**Watts-Strogatz small-world model** Start with a lattice with \( n = m^2 \) nodes. Each node \( u \) is connected to its neighbors:

\[
\{w \in V - \{u\} : ||u - w|| = 1\}
\]

Where \( || \cdot || \) is \( L_1 \) (taxi-cab) distance. These represent strong ties (close friends) in a network. Each node chooses a long-range ”shortcut” uniformly at random. In other words, each node will have a weak tie to another node in that lattice, and that will be chosen at random. So if \( u \rightsquigarrow v \) denotes a shortcut between nodes \( u \) and \( v \), we have

\[
\mathbb{P}(u \rightsquigarrow v) = \frac{1}{|V - \{u\}|} = \frac{1}{n - 1}
\]

Now let \( T_{alg}(u, v) \) be the time it takes to route from \( u \) to \( v \) given some algorithm \( alg \). We want to know if \( W - S \) is algorithmically small world. In other words, is \( \mathbb{E}[T_{alg}(u, v)] = O(\log(n)) \)?
Theorem 9.1. For "most" $u, v \in V$ and any decentralized algorithm $\text{alg}$

$$E[T_{\text{alg}}(u, v)] = \Omega(m^{\frac{2}{3}})$$

So then we have that W-S is not algorithmically small-world. We thus want to modify it so that we have $O(\log(n))$.

9.2 Kleinberg small-world model

Consider an infinite family parametrized by $\alpha \geq 0$. We will have nodes choose a long-range "shortcut" with probability:

$$P(u \leftrightarrow v) = \frac{1}{||u-v||^\alpha} \sum_{w \neq v} \frac{1}{||u-w||^\alpha}$$

We call this the "Kleinberg Model"

Note that $\alpha$ is a clustering parameter for weak ties. Also, if $\alpha = 0$, we get the W-S small world model, while if $\alpha = \infty$, we get $P(u \leftrightarrow v) = \frac{1}{N(u)}$, where $N(u)$ denotes the set of neighbors of $u$.

9.2.1 Choosing $\alpha$

We want a model that is algorithmically small-world. How can we choose $\alpha$ so that the Kleinberg Model is algorithmically small-world?

**Idea** When $\alpha$ is too small, weak ties spread too thin, but when $\alpha$ gets large, the shortcuts don’t make a difference.

We want to find $\alpha \in (0, \infty)$ such that we get the the algorithmic small world property.

**Theorem 9.2.** If $\alpha = 2$, then with the greedy algorithm,

$$E[T_{\text{greedy}}(u, v)] = O(\log(n))$$

**Proof:** idea: Pick some target $v$. We take a box around it of size $2s$, so there are $2s$ nodes in the box. Then if $u \notin$ box, we find how long it takes on average to get to $\frac{||u-v||}{2^j}$, $j = 1, 2, ..., \log_2(m)$. \[ \square \]

**Theorem 9.3.**

i) If $\alpha \in [0, 2)$, then for "most" pairs $(u, v)$, (proportion of pairs for which this holds goes to 1 as $n \to \infty$), and any decentralized algorithm $\text{alg}$,

$$E[T_{\text{alg}}(u, v)] = \Omega(m^{\frac{2-\alpha}{3}})$$

ii) If $\alpha > 2$,

$$E[T_{\text{alg}}(u, v)] = \Omega(m^{\frac{\alpha-2}{\alpha-1}})$$
Proof: Consider
\[ z = \sum_{w \neq u} \frac{1}{||u-w||^\alpha} = \sum_{i=1}^{2m} \sum_{w: ||u-w||=i} \frac{1}{i^\alpha} \sim \sum_{i=1}^{2m} i \cdot i^\alpha \]

Now
\[ \int_1^m x \cdot x^{-\alpha} dx = \int_1^m x^{1-\alpha} dx \sim m^{2-\alpha} \text{ when } \alpha \neq 2 \text{ and } \log(m) \text{ when } \alpha = 2 \]

Idea for (i): Choose some destination \( v \) and consider a box of side length \( m^\beta \) for some given \( \beta < 1 \). Take an \( m^\beta \) box around \( v \). Then we have that the proportion of sources \( u \in V \) such that \( u \) is inside the box is \( \frac{m^{2\beta}}{m^2} = m^{2(\beta-1)} \Rightarrow \) as \( n \to \infty \), the proportion of such sources \( u \to 0 \).

Now consider \( u \in V \) outside of the box. Without any shortcuts, we'll need to take at least \( \Omega(m^\beta) \) steps to get into the box. Then if \( P \) is the probability that \( u \) has a shortcut into the box,

\[ P \leq \frac{\text{sum}_{w\in\text{box}} \frac{1}{||u-v||^\alpha}}{z} \leq \frac{\text{number of nodes in box}}{z} = \frac{m^{2\beta}}{z} \sim \frac{m^{2\beta}}{m^{2-\alpha}} = m^{2\beta-2+\alpha} \]

\( \Rightarrow \) On average, we will have to check \( m^{-2\beta+2-\alpha} \) nodes to find a shortcut into the box. Then the minimum amount of time to route from \( u \) to \( v \geq \Omega\{\max\{m^\beta, m^{-2\beta+2-\alpha}\}\} \). We want to minimize this over \( \beta \), which we can do by finding \( \beta \) such that \( \beta = -2\beta + 2 - \alpha \):

\[ \beta = -2\beta + 2 - \alpha \Rightarrow 3\beta = 2 - \alpha \Rightarrow \beta = \frac{2 - \alpha}{3} \]

Then we have that \( \mathbb{E}[T_{alg}(u, v)] = \Omega(m^{\frac{2-\alpha}{3}}) \). \( \square \)