1. Let $V$ be an $n$-dimensional vector space and $W$ be an $m$-dimensional vector space.
   a) Suppose $n < m$. Show that there is no linear transformation $L: V \to W$ such that $L$ is onto.
   b) Suppose $n > m$. Show that there is no linear transformation $L: V \to W$ such that $L$ is one-to-one.
   c) Prove that $L: V \to W$ is an isomorphism only if $n = m$.
   d) Let $U_3$ be the space of $3 \times 3$ upper triangular matrices and define the linear transformation $L: U_3 \to \mathcal{M}_{33}$ by $L(A) = \frac{1}{2}(A + A^T)$. Is $L$ onto? Is $L$ one-to-one?

2. Let $V = \mathcal{P}_2$ with standard basis $B = \{1, x, x^2\}$ and let $W = \mathcal{M}_{22}$ with standard basis $D = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$. Consider $L: V \to W$ given by
   
   \begin{align*}
   L(p) &= \begin{bmatrix}
   p(1) - p(0) & p(2) - p(0) \\
   p(-1) - p(0) & p(-2) - p(0)
   \end{bmatrix}.
   \end{align*}

   (For example, $L(x^2) = \begin{bmatrix} 1^2 - 0^2 & 2^2 - 0^2 \\
   (-1)^2 - 0^2 & (-2)^2 - 0^2 \end{bmatrix}$.)
   a) Prove that $L$ is a linear transformation.
   b) Find the matrix representation $[L]_{D,B}$ of $L$ with respect to the bases $B$ and $D$.
   c) What is the dimension of $\ker(L)$? Find a basis for $\ker(L)$.
   d) What is the dimension of $\text{range}(L)$? Find a basis for $\text{range}(L)$.
   e) Verify the dimension theorem (i.e., rank-nullity theorem) for $L$.

3. Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 5 & 7 \\ 5 & 10 & 13 & 18 \end{bmatrix}$ so that $\text{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.
   b) Find a basis for the span of the four vectors in part (a).
4. Let \( L : \mathbb{R}^2 \to \mathbb{R}^2 \) be given by \( L \left( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} 8x_1 - 10x_2 \\ 3x_1 - 3x_2 \end{bmatrix} \). Define the standard basis \( B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \) and an alternate basis \( D = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \end{bmatrix} \right\} \). Consider a vector \( v = \begin{bmatrix} 8 \\ 3 \end{bmatrix} \).

a) Find the change of basis matrices \( P_{DB} \) (i.e., from \( B \) to \( D \)) and \( P_{BD} \) (i.e., from \( D \) to \( B \)).

b) Compute \( [v]_B \) and \( [v]_D \).

c) Find \( [L]_BB \) and \( [L]_{DD} \).

5. Consider \( A = \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix} \).

a) Write the characteristic polynomial \( p_A(\lambda) \) and use this to determine the eigenvalues of \( A \).

[Hint: Factor out a term of the form \((\lambda - 1)\).]

b) Find the eigenspaces corresponding to the eigenvalues of \( A \).

c) Is \( A \) diagonalizable? If so, find a nonsingular matrix \( P \) and a diagonal matrix \( D \) such that \( A = PDP^{-1} \).

6. Let \( A = \begin{bmatrix} -8 & 4 & -3 & 2 \\ 2 & 1 & -1 & 0 \\ -3 & -5 & 4 & 0 \\ 2 & -4 & 3 & -1 \end{bmatrix} \).

a) Find the determinant of \( A \) using a cofactor expansion.

b) Find the determinant of \( A \) by using row operations to put \( A \) into upper triangular form. Verify that your answer agrees with part (a).

c) Is \( A \) nonsingular (i.e., invertible)?

7. The following two questions are unrelated to each other.

a) Show that \( V = \mathbb{R} \) with the usual operation of scalar multiplication but with addition given by \( x \oplus y = 2(x + y) \) is not a vector space.

b) Consider the subset \( S \) of all matrices in \( M_{5 \times 5} \) which have eigenvalue 1. Is \( S \) a subspace of \( M_{5 \times 5} \)? Explain why or why not.

8. True or false? Explain your answers.

a) The plane \( x_1 + 3x_2 - 4x_3 = 1 \) is a subspace of \( \mathbb{R}^3 \).

b) If \( A \) is a \( 3 \times 5 \) matrix, then \( \dim(\text{Ker}(A)) \geq 2 \).

c) Let \( B = \{b_1, \ldots, b_n\} \) be a basis for a vector space \( V \). If \( n \) vectors \( \{d_1, \ldots, d_n\} \) span \( V \) then the coordinate vectors \( \{[d_1]_B, \ldots, [d_n]_B\} \) are linearly independent.
d) Every linear transformation $L: \mathbb{R}^5 \to \mathbb{R}^4$ takes the form $L(x) = Ax$ with $A$ a $5 \times 4$ matrix.

e) Let $\text{rref}(A)$ be the reduced row-echelon form of a matrix $A$. Then, the pivot columns of $\text{rref}(A)$ form a basis of the column space of $A$ (i.e., the span of the columns of $A$).

f) The vectors $b_1 = 1 + t + 2t^2$, $b_2 = 2 + 3t + 5t^2$, $b_3 = 3 + 7 + 9t^2$ form a basis for $\mathcal{P}_2$.

g) [Harder...] The equation $p''(t) - p(t) = q(t)$ has a solution $p \in \mathcal{P}_3$ for any $q \in \mathcal{P}_3$. 