1. Let \( A = \begin{bmatrix} 4 & 3 & 1 & 2 \\ 1 & 9 & 0 & 2 \\ 8 & 3 & 2 & -2 \\ 4 & 3 & 1 & 1 \end{bmatrix} \).

a) Calculate the determinant of \( A \) using a cofactor expansion.

**Solution:** We expand \( \det(A) \) about the third column:

\[
\det(A) = 1 \cdot \begin{vmatrix} 8 & 3 & -2 \\ 4 & 3 & 1 \end{vmatrix} + 2 \cdot \begin{vmatrix} 1 & 9 & 2 \\ 4 & 3 & 1 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 9 & 2 \\ 8 & 3 & 2 \end{vmatrix} = -111 - 66 + 180 = 3.
\]

b) Recalculate the determinant using row reduction to verify your answer to (a).

**Solution:** To calculate the determinant, we can put \( A \) into upper triangular form using row operations as follows:

\[
A = \begin{bmatrix} 4 & 3 & 1 & 2 \\ 1 & 9 & 0 & 2 \\ 8 & 3 & 2 & -2 \\ 4 & 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 9 & 0 & 2 \\ 4 & 3 & 1 & 2 \\ 8 & 3 & 2 & -2 \\ 4 & 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 9 & 0 & 2 \\ 0 & -33 & 1 & -6 \\ 0 & -3 & 0 & -6 \\ 0 & 0 & 0 & -1 \end{bmatrix} = U.
\]

Therefore, \( 3 = \det(U) = (-1) \times (-1) \times \det(A) \) so \( \det(A) = 3 \) as expected.

c) What is the determinant of \(-2A\)? Why?

**Solution:** \( \det(-2A) = (-2)^4 \det(A) = 16 \cdot 3 = 48 \) since \( A \) has 4 rows.

2. Prove that if \( A \) is an orthogonal matrix (i.e., \( A^T = A^{-1} \)) then the determinant of \( A \) is either 1 or \(-1\).

**Solution:** Since

\[
\det(A) = \det(A^T) = \det(A^{-1}) = \frac{1}{\det(A)}
\]

we have that \( (\det(A))^2 = 1 \), so \( \det(A) = \pm 1 \).

3. Let \( A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix} \).

a) Determine the eigenvalues of \( A \).

**Solution:** The characteristic polynomial is

\[
p_A(\lambda) = \det(A - \lambda I) = -\lambda^3 + \lambda = -\lambda(\lambda + 1)(\lambda - 1)
\]
so the eigenvalues are \( \lambda = 1, -1, 0 \).

b) Find a nonsingular matrix \( P \) and a diagonal matrix \( D \) such that \( A = PDP^{-1} \).

**Solution:** Computing the eigenspaces for each eigenvalue and putting the corresponding fundamental eigenvectors as the columns of a matrix \( P \), we find that \( A = PDP^{-1} \) with

\[
P = \begin{bmatrix} -1 & -1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]

c) Compute the determinant of \( A \) only using your answer to part (a) (i.e., do not compute the determinant directly).

**Solution:** \( \det(A) = p_A(0) = 0 \).

4. The parts of the following question are unrelated.

a) Is \( V = \mathbb{R} \) with the usual scalar multiplication, but with addition defined as \( x \oplus y = 3(x + y) \) a vector space? Justify your answer.

**Solution:** No. The operation \( \oplus \) is not associative since \((x \oplus y) \oplus z = 3(3(x + y) + z) = 9x + 9y + 3z \neq 3x + 9y + 9z = 3(x + 3(y + z)) = x \oplus (y \oplus z)\).

b) Find the zero vector and the additive inverse of the vector space \( \mathbb{R}^2 \) with operations \([x, y] \oplus [w, z] = [x + w + 3, y + z - 4] \) and \( a \odot [x, y] = [ax + 3a - 3, ay - 4a + 4] \).

**Solution:** \( 0 = 0 \odot [x, y] = [0x + 3(0) - 3, 0y - 4(0) + 4] = [-3, 4] \) while \(-([x, y]) = [-x - 6, -y + 8] \).

c) If \( V \) is a vector space with subspace \( W_1 \) and \( W_2 \), prove that \( W_1 \cap W_2 \) is also a subspace.

**Solution:** Since the subspaces \( W_1 \) and \( W_2 \) both contain the zero vector, \( 0 \in W_1 \cap W_2 \) and \( W_1 \cap W_2 \) is nonempty. Now suppose \( x, y \in W_1 \cap W_2 \) and \( c \) is a scalar. Then \( x, y \in W_1 \) and \( x, y \in W_2 \) so \( x + y \in W_1 \) and \( x + y \in W_2 \) since \( W_1 \) and \( W_2 \) are closed under vector addition. Therefore, \( x + y \in W_1 \cap W_2 \) and \( W_1 \cap W_2 \) is closed under vector addition as well. Similarly we find \( W_1 \cap W_2 \) is closed under scalar multiplication, so \( W_1 \cap W_2 \) is a subspace.


a) Is \( S \) linearly independent? If not, find a maximal linearly independent subset.

**Solution:** Let \( A = \begin{bmatrix} 2 & 2 & 7 \\ -3 & 9 & 6 \\ 4 & -12 & -10 \\ -1 & 3 & 4 \end{bmatrix} \) be the matrix whose columns are vectors in \( S \). Then \( \text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \), which does not have a pivot in each column so \( S \) is not linearly independent. One maximal linearly independent subset consists of the pivot columns of \( A \)—i.e., \( B = \{ [2, -3, 4, -1]^T, [3, 1, -2, 2]^T, [2, 8, -12, 3]^T \} \).
b) Does $S$ span $\mathbb{R}^4$? If not, express $\text{span}(S)$ in terms of a minimal spanning set.

**Solution:** No, $S$ does not span $\mathbb{R}^4$ since $\text{rref}(A)$ does not have a pivot in every row. A minimal spanning subset of $S$ is the set $B$ found in part (a), and $\text{span}(S) = \text{span}(B)$.

c) Construct a basis for $\text{span}(S)$. What is $\dim(\text{span}(S))$?

**Solution:** $B$ forms a basis for $\text{span}(S)$, and $\dim(\text{span}(S)) = |B| = 3$.

d) Construct a basis for $\mathbb{R}^4$ that contains the maximal linearly independent subset found in part (a).

**Solution:** We must extend the linearly independent set $B$ by adding to it another vector that is linearly independent to $B$. For example, let $v = [1, 0, 0, 0]^T$ and define $\bar{B} = B \cup \{v\}$. Putting the vectors in $\bar{B}$ as columns of a matrix $\bar{A}$ we find that $\text{rref}(\bar{A}) = I_4$ so $\bar{B}$ is a basis of $\mathbb{R}^4$.

6. Prove that all vectors orthogonal to $[2, -3, 1]^T$ forms a subspace $W$ of $\mathbb{R}^3$. What is $\dim(W)$ and why?

**Solution:** Let $v = [2, -3, 1]^T$. Note that $0 \in W$ since $0 \cdot v = 0$ so $W$ is nonempty. Now suppose $x, y \in W$ and $c$ is a scalar. Then $(x + y) \cdot v = (x \cdot v) + (y \cdot v) = 0 + 0 = 0$ and $(cx) \cdot v = c(x \cdot v) = c0 = 0$.

We will compute $W$ explicitly in order to find its dimension. Since $x = [x_1, x_2, x_3]^T \in W$ if and only if $[2, -3, 1]^T \cdot x = 2x_1 - 3x_2 + x_3 = 0$, we have that $x_3 = -2x_1 + 3x_2$ so $x = x_1[1, 0, -2]^T + x_2[0, 1, 3]^T$. Therefore, $B = \{[1, 0, -2]^T, [0, 1, 3]^T\}$ is a basis for $W$ and $\dim(W) = 2$. 

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