M365G Homework, due May 1, 2012

1. Let \( S_1 \) be a surface and \( p \) be a point on that surface. Show that there is a direct isometry of \( \mathbb{R}^3 \) that sends \( p \) to the origin and that sends a neighborhood of \( p \) in \( S_1 \) to a surface \( S_2 \) of the form \( z = f(x, y) \), where
   \[
   f(x, y) = ax^2/2 + by^2/2 + O(r^3),
   \]
   with \( O(r^3) \) meaning terms that go to zero at least as fast as \( (x^2 + y^2)^{3/2} \) as \( x, y \to 0 \). More precisely, it means that \( |f(x, y) - (ax^2/2 + by^2/2)|/(x^2 + y^2)^{3/2} \) is bounded in a neighborhood of the origin. [Note: since everything is smooth, there is a Taylor series for \( f(x, y) \). The expression \( O(r^3) \) describes all the terms that go as \( x^i y^j \) with \( i + j \geq 3 \). This also means that the derivatives of the \( O(r^3) \) terms are \( O(r^2) \), and that the second derivatives are \( O(r) \).]

   In the rest of this problem set, your answers should all be of the form
   (Some quantity) = (Some expression involving \( a, b, x, y \)) + \( O(r^{\text{some power}}) \).
   Don’t forget that \( (1 + \epsilon)^n = 1 + n\epsilon + O(\epsilon^2) \). This is particularly useful for \( n = 1/2 \) and \( n = -1 \).

2. Using coordinates \( u = x \) and \( v = y \), find expressions for the first and second fundamental forms of \( S_2 \) as a function of \( x, y \), and compute the Gauss curvature \( K \).

3. Compute all the Christoffel symbols for \( S_2 \) (see Prop 7.4.4), and compute the commutator \( [\nabla_1, \nabla_2] \). Your answer should be a \( 2 \times 2 \) matrix, from which you can infer the value of \( C_1 \) (as defined in class).

4. Show that the Gauss equations (Prop. 10.1.2, not to be confused with the Gauss equations of Prop 7.4.4 – Gauss had a lot of equations!) apply to \( S_2 \) at the origin. An earlier version of this problem also asked about the Codazzi-Mainardi equation. Do NOT evaluate those, as the expressions depend strongly on the \( O(r^3) \) terms in \( f(x, y) \). Also, this is practically the same calculation as problem 3. Either problem is enough to conclude that \( K = C_1 \) at the origin.

5. Returning to the original surface \( S_1 \), show that \( C_1(p) = K(p) \). Conclude that \( C_1 \) and \( K \) are the same geometric quantity for all points on all surfaces.

6. Geodesics on \( S_2 \) are approximated very well by intersections of \( S_1 \) with vertical planes through the origin. That is, the shortest path from \((0, 0, 0)\) to \((x_0, y_0, z_0)\) has \( x/y \) constant. Taking this result for granted, compute the distance from the origin to \((x, y, f(x, y))\).

7. Using the results of Problem 6, construct geodesic normal coordinates around the origin of \( S_2 \).
8. Now consider the “circle” obtained by fixing a value of $r$ in the geodesic normal coordinates. [Note that in this context $r$ is the geodesic distance from the origin, which is NOT the same as $\sqrt{x^2 + y^2}$. This isn’t a repeat of an exam problem!] Show that the circumference of that “circle” is $2\pi r (1 - ab r^2/6) +$ higher order. This shows that the defect in the circumference is proportional to the Gauss curvature.

9. Finally, compute the area enclosed by the circle [hint: $\int$ (circumference) $dr$] and the isoperimetric ratio $\text{Area}/(\text{circumference})^2$.

Your answer to problem 9 SHOULD match the results of the exam problem. In the exam, we considered the intersection of a cylinder $x^2 + y^2 = r^2$ with the surface. That resulted in a curve that approximates the “circle” of problems 8 and 9. That approximation wasn’t good enough to compute the circumference and area individually, but WAS good enough to compute the isoperimetric ratio. This is because a circle maximizes the ratio, so deviations from that circular shape only affect the ratio to second order.