The goal of this homework is to learn all about determinants (Section 3). We then apply this to eigenvalues and eigenspaces as well as to linear transformations.

**Finish Chapter 5. Read Section 3 and do the following ten exercises on it:**
We will use the following theorem from class (cf. Theorem 3.2):

**Theorem 1.** There is a unique function $\det : M_{nn} \to \mathbb{R}$, called the determinant, such that

1. $\det(I) = 1$, and
2. If $A'$ is obtained from $A$ by an elementary row operation, then
   - (I) If $A'$ is obtained by multiplying a row by $\lambda \neq 0$, then $\det(A') = \lambda \det(A)$;
   - (II) If $A'$ is obtained by adding $c$ times the $i$-th row to the $j$-th row, then $\det(A') = \det(A)$;
   - (III) If $A'$ is obtained from $A$ by swapping two rows, then $\det(A') = -\det(A)$.

(We said also $\det(AB) = \det(A) \det(B)$, but this is not necessary for the uniqueness.)

1. Section 3.2, problem 1.
2. Section 3.2, problem 2.

In the next two exercises, we prove Theorem 3.2 using only the above stated theorem (not using Section 3.1).

3. Using only the above stated theorem, we prove the following. Suppose you have an upper-triangular matrix with nonzero diagonal entries. Prove that the determinant is the product of the diagonal entries. Hint: using type I operations, show that the determinant is the product of the diagonal entries and the determinant of an upper-triangular matrix with ones on the diagonal. Using type II row operations, show that the latter determinant equals one (recall that the determinant of the identity is one from our theorem).

4. Prove the following statement we mentioned in class: a matrix has nonzero determinant if and only if it is invertible, using again only the aforementioned theorem from class. Hint: Using Gaussian elimination and row operations, show that $\det A$ is a nonzero multiple of $\det A'$ for $A'$ the reduced row-echelon form matrix obtainable from $A$ by row operations. If $A$ is invertible, then $A' = I$, so conclude that $\det A$ is nonzero. If $A$ is not invertible, then $A'$ has a row which is zero. Using a type I row operation on the zero row (multiply it by any $c \neq 0$), show that $\det A' = c \det A'$ for all $c \in \mathbb{R}$. For this to be possible, conclude that we must have $\det A' = 0$.

5. The goal of this problem is to prove that $\det(A) = \det(A^t)$ for an arbitrary matrix. Along the way we compute determinants of matrices corresponding to elementary row operations.

Recall the matrices $R$ such that $A \mapsto RA$ is a row operation, from class a long time ago. Namely, given a row operation, $R$ is obtained from $I$ by applying the same row operation. In more detail, in type I, $R$ is a diagonal matrix with all entries equal to 1 but the $i$-th one, which is $\lambda$ (this corresponds to multiplying the $i$-th row by $\lambda$. Next, in type II, $R$ is a matrix which has ones on the diagonal, $c$ in the $ji$-th position, and zero elsewhere (corresponding to adding $c$ times the $i$-th row to the $j$-th row). Finally, in type III, $R$ is a matrix which is obtained from the identity matrix by swapping first the $i$-th and $j$-th rows, then the $i$-th and $j$-th columns (this is the...
matrix which corresponds to swapping the $i$-th and $j$-th rows under the operation $A \mapsto RA$).

(i) Using the theorem above, compute the determinant of $R$ in all of the three cases.
(ii) Show that this is the same as the determinant of the transpose $R^t$, and moreover that $R^t$ is a matrix of a different elementary row operation.
(iii) Use this to prove that $\det(A) = \det(A^t)$. Hint: first note that $A$ is invertible if and only if $A^t$ is invertible (either because $BA = I = AB$ implies $A^tB^t = I = B^tA^t$, or by recalling that $A$ is invertible if and only if its rank is $n$ and that rank of $A$ equals rank of $A^t$.) If $A$ is not invertible we then get $\det(A) = \det(A^t) = 0$ by the preceding problem. Next suppose $A$ is invertible. Recall that, in this case, $A$ is obtained from the identity by row operations: $A = R_1 \cdots R_nI = R_1 \cdots R_n$ where each $R_i$ is a matrix of the above form. Then, we get $\det(A) = \det(R_1) \cdots \det(R_n)$, either from the theorem above, or from $\det(XY) = \det(X) \det(Y)$. Similarly, using (ii), we get $\det(A^t) = \det(R_n^t) \cdots \det(R_1^t)$, which is the same as $\det(R_1) \cdots \det(R_n)$ by (ii) again. Conclude the statement.

In the next couple problems, we will use determinants to compute eigenvalues. Let $A \in \mathcal{M}_{nn}$. Note that, for $c \in \mathbb{R}$, the kernel $\ker(cI - A)$ is the $c$-eigenspace of $A$, i.e., the subspace of eigenvectors of eigenvalue $c$ (together with the zero vector).

(6) Show that the following are equivalent:
(a) The $c$-eigenspace is nonzero;
(b) $cI - A$ is not invertible;
(c) $\det(cI - A) \neq 0$.

Hint: Recall that $cI - A$ is invertible if and only if its rank is $n$. Now by the dimension theorem, $\rk(cI - A) = n - \dim \ker(cI - A)$. Conclude that $\ker(cI - A) \neq 0$ if and only if $\rk(cI - A) < n$, i.e., if and only if $cI - A$ is not invertible. Now use the fact we proved in a preceding problem, linking this with the determinant of $cI - A$.

As a consequence of the above problem, the eigenvalues of $A$ are exactly the roots of the polynomial $\det(xI - A)$, called the characteristic polynomial in §3.4 (cf. Theorem 3.14).

(7) Section 3.4, Exercise 1. **Moreover, find all roots of the polynomials: these are the eigenvalues.** Note: for (e), you might want to follow the book’s suggestion, and in this case you will want to read Section 3.1.

(8) Section 3.4, Exercise 2. Here, $E_\lambda = \ker(\lambda I - A)$ is the $\lambda$ eigenspace. You don’t have to learn what fundamental eigenvectors are (which is explained in Section 3.4): it is fine if you replace “fundamental eigenvectors” by any basis for $E_\lambda$ (Section 3.4 comes before you learn what a basis is).

(9) Prove the following: if $T : V \to V$ is any linear transformation, then the characteristic polynomial $\det(xI - [T]_{BB})$ does not depend on the choice of basis $B$. Conclude that the eigenvalues of $[T]_{BB}$ also do not depend on $B$. Hint: This is just like the proof we did of invariance of $\det([T]_{BB})$ itself, since $P(xI)P^{-1} = xI$ for all invertible $P$.

This motivates:

**Definition 2.** For $T : V \to V$ a linear transformation, $\chi_T(x) := \det(xI - [T]_{BB})$ is called the characteristic polynomial, independent of $B$. 

(10) Recall that, for $T : V \to V$ a linear transformation, the eigenvalues of $T$ are the $\lambda \in \mathbb{R}$ such that, for some nonzero $v \in V$, $T(v) = \lambda v$. Show that the eigenvalues of $T$ are the same as the eigenvalues of $[T]_{BB}$ for any $B$. Conclude from the previous problem that the eigenvalues of $T$ are the roots of the characteristic polynomial $\chi_T(x)$. 