Lecture 12

Span and dimension: Given a set \( \{v_1, \ldots, v_m\} \) of vectors in a vector space \( \mathbb{R}^n \),

Definition \( \text{Span}(v_1, \ldots, v_m) = \{c_1 v_1 + \cdots + c_m v_m\} \)

HW: You prove: \( \text{Span}(v_1, \ldots, v_m) \subseteq \mathbb{R}^n \cap \{c_1 v_1 + \cdots + c_m v_m\} \)
is the intersection of all vector subspaces containing \( v_1, \ldots, v_m \).

Definition A vector subspace \( W \subseteq V \) is a subset which is a vector space under \( +, \cdot \) (addition, scalar mult in \( V \)).

[Ex: Row space \( \leq \mathbb{R}^n \), eigenspace \( \leq \mathbb{R}^n \), solution space of \( Ax = 0 \) \( \leq \mathbb{R}^n \)]

Defn The dimension of a vector space \( V \) is the minimum \( m \geq 0 \) such that \( V = \text{Span}(v_1, \ldots, v_m) \) for some \( v_1, \ldots, v_m \in V \).

Example \( \text{dim}(\mathbb{R}^3) = \text{Span}(\emptyset) \Rightarrow \text{dim}(\emptyset) = 0 \)

This is the only 0-dimensional vector space.
Example: Dimension of row space: (21)

Recall that Row space is defined as:

\[
\text{row space} = \text{Span (rows of } A) = \text{span (v₁, ..., vₘ)} \quad (\text{v₁, ..., vₘ are rows of } A).
\]

\[
\implies \dim (\text{row space}) \leq m.
\]

In fact, \( \dim (\text{row space}) = \text{rank} \):

Because \( \text{row space } (A) = \text{row space of reduced row-echelon form matrix } B \) obtained by row ops

\[
\text{row space } \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

can be expressed using \# parameters = \# nonzero rows

these parameters are independent, \( = \text{rank} \).

Example:

\[
\begin{pmatrix}
1 & 2 & 3 & 0 & 5 \\
0 & 1 & 0 & 4 & 0
\end{pmatrix}
\]

\[
\text{row space} = \text{Span } [1 \; 2 \; 3 \; 0 \; 5], \; [0 \; 0 \; 0 \; 1 \; 4]
\]

\[
= \begin{pmatrix}
t \; , \; 2t \; , \; 3t \; , \; u \; , \; 5t + 4u
\end{pmatrix}
\]

\[\text{two free parameters.} \]

\( \Rightarrow \text{need } \geq 2 \text{ vectors to span row space.} \)
Key definition:

A vector space is a set $V$ equipped with vector addition (+), a scalar multiplication, and a zero vector $0 \in V$ satisfying:

**Addition:**
- $v + w = w + v$ (commutativity)
- $(u + v) + w = u + (v + w)$ (associativity)
- $0 + v = v = v + 0$ (add. id.)
- For every $v$, there exists (add. inv.) $-v$ s.t. $v + (-v) = 0 = (-v) + v$.

**Scalar mult.:**
- $a(bv) = (ab)v$ (associativity)
- $1v = v$ (mult. id.)

**Distributivity:**
- $a(v + w) = av + aw$, $a \in \mathbb{R}$
- $v + w = v$, $v, w \in V$.
- $(a + b)v = av + bv$.

Important! otherwise you could define $v = 0 \forall v \in V$.

Point: These are properties that are satisfied in our motivating examples.

Using this definition, we prove theorems that hold for all examples (satisfying these properties).
NOTE: In book: beginning of §4.1, there is a variant of the definition: it also includes 2 axioms:

\[
\begin{align*}
\text{(A)} & \quad u + v \in V \quad \forall u, v \in V \\
\text{(B)} & \quad a \cdot v \in V \quad \forall a \in \mathbb{R}, \: v \in V.
\end{align*}
\]

I said: Vector space equipped with set of vectors, scalar multiplication.
Book is a bit more redundant.

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Examples:
\[\mathbb{R}^n\] satisfies these properties.

What about new spaces, eigenspaces, solution space, column space?

There are all subsets of \[\mathbb{R}^n\].

So showing they are vector spaces is the same as showing they are subspaces (using same addition, mult, 0).

Definition: A vector subspace \(W \subseteq V\) is a subset of a vector space \(V\) s.t.:

(i) \(0 \in W\), (ii) \(w_1, w_2 \in W \Rightarrow w_1 + w_2 \in W\)

(iii) \(c \in \mathbb{R}, \: w \in W \Rightarrow c \cdot w \in W\)

(iv) The axioms of vector space on \(W\) are satisfied.
Theorem 12 A subset $W \subseteq V$ is a subspace if and only if:
(a) it is nonempty
(b) $w_1, w_2 \in W \Rightarrow w_1 + w_2 \in W$
(c) $w \in W, c \in \mathbb{R} \Rightarrow cw \in W$

Proof. If $W \subseteq V$ is a subspace, $0 \in W \Rightarrow$ nonempty, other conditions of theorem are immediate.

Conversely, suppose $W \subseteq V$ is a subset satisfying (a), (b), (c). (h) = (ii), (e) = (iii), (f).

Prove (i):

Proposition $0 \cdot v = 0 \forall v \in V$.

By proposition, if $v \in W$ is any vector, $0 \cdot v \in W$.

\[ \therefore (i). \]

Proof of proposition: Here's a trick:

\[ 0 \cdot v = (0 + 0) \cdot v = 0 \cdot v + 0 \cdot v \]

Subtract $0 \cdot v$ from both sides:

\[ 0 \cdot v + (-0 \cdot v) = (0 \cdot v + 0 \cdot v) + (-0 \cdot v) \]

LHS = 0 by additive inverse,

RHS = $0 \cdot v + (0 \cdot v - 0 \cdot v) = 0 \cdot v + 0 = 0 \cdot v$

\[ \therefore 0 \cdot v = 0 \]
Finally we need to show (iv) axioms of (61) vector space: are satisfied for $W \subseteq V$.

These are immediate because we can compute addition, scalar mult. in $W$ or $V$ and get same answer.

$\therefore W$ is a vector subspace $\Rightarrow$ it's a vector space.

**Corollary:** row space, eigenspaces, solution spaces of $A^T x = 0$ and column space are vector subspaces of $\mathbb{R}^n$.

**Proof:** for row space: $0$ vector is always in row space (or any particular row is in row space) $\Rightarrow$ nonempty.

Why closed under $+$:

\[(c_1v_1 + \ldots + c_m v_m) + (d_1v_1 + \ldots + d_m v_m)\]

\[v_i = \text{rows} \]

\[= (c_1+d_1)v_1 + \ldots + (c_m+d_m)v_m.\]

Why closed under scalar mult:

\[a(b_1v_1 + \ldots + b_m v_m)\]

Distribute:

\[= a(b_1v_1) + \ldots + a(b_m v_m)\]

\[= (ab_1)v_1 + \ldots + (ab_m)v_m \vee\]
Comment: The same argument shows:

\[ \text{Span}(V_1, \ldots, V_m) \text{ is a vector subspace} \]

\[ \forall v_1, \ldots, v_m \in V, \quad \forall v \in V \text{ vector space}. \]

HW: \[ \text{Span}(V_1, \ldots, V_m) = \text{intersection of} \]

\[ \forall v_1, \ldots, v_m \]

More examples of vector spaces:

Functions: \( \mathbb{R} \rightarrow \mathbb{R} \)

we need to define:

- \( \mathcal{C} \) function
- Sum of functions
- Scalar mult. of f in \( \mathcal{C} \)

0 function = the function \( f \) s.t. \( f(x) = 0 \) \( \forall x \)

Graph:

\[ f(x) = 0 \]

Addition: \( f + g \) is the function

\[ (f + g)(x) = f(x) + g(x) \]

Scale mult: \( cf \) \( (x) = c(f(x)) \), \( c \in \mathbb{R} \)

Proposition: This forms a vector space: all 8 axioms satisfied

(Proof is by evaluating at every \( x \in \mathbb{R} \)).
Example: \( \mathbb{R}^n \setminus \{0\} \)

0 is not a subset \( \Rightarrow \) not subspace.

Also not closed under addition, scalar multiplication.

Thus are either closed under +, not -
or vice versa.