Lecture 25

Plan: Two more lectures, state two fundamental theorems from lin alg:
(Gram-Schmidt: today)

- Spectral theorem for normal operators
- Polar decomposition (gen. of polar coords)

\[ z = r e^{i\theta} \]

\[ r (\cos \theta + i \sin \theta) \]

Polar coords on \( \mathbb{C} = \mathbb{R}^2 \)

Last time: observed (informally):

**Defn** A vector space \( V \) is the direct sum of \( u_1, u_2 \in V \) if \( u_1 + u_2 \rightarrow \exists! u_1, u_2 \in V \), s.t. \( v = u_1 + u_2 \). Write: \( V = u_1 \oplus u_2 \)

**Thm** \( F^n = V \oplus V^\perp \forall V \subseteq F^n \).

Proved Thm if \( V \) admits an orthonormal basis.

Today: Prove Thm on basis of \( V \), construct it!
(Gram-Schmidt)
Examples of direct sum:

First, define \( V = V_1 \oplus \ldots \oplus V_k \) if \( \forall v \in V \)

\[ \exists! \ v_i \in V_1, \ldots, v_k \in V_k \text{ s.t.} \]

\[ V = v_1 + \ldots + v_k. \]

**Example**: If \( V_1, \ldots, V_n \) is a basis of \( V \),

\[ V = V_1 \oplus \ldots \oplus V_n, \quad V_j = \text{span}(V_j) \]

**Claim** (i.e. \( \dim V_j = 1 \))

Why? \( \forall v \in V, \exists! \ a_1, \ldots, a_n \) s.t.

\[ v = a_1 v_1 + \ldots + a_n v_n. \] That means

\( a_1 v_1, \ldots, a_n v_n \) are the unique vectors in \( V_1, \ldots, V_n \) which sum to \( v \).

Conversely, if \( V = V_1 \oplus \ldots \oplus V_n \) \( \dim V_j = 1 \)

\( \implies \) we can get a basis of \( V \) from any

nonzero vectors \( v_i \in V_1, \ldots, v_n \in V_n \).

**Example**: If \( T : V \rightarrow V \) is an operator which

admits an eigenbasis, then:

\[ V = \bigoplus_{k=1}^{n} E_k. \]

\( \exists! \ \lambda \) distinct eigenvalues

\( E_{\lambda_1}, \ldots, E_{\lambda_n} \) eigenspaces.

AND conversely
Why true? If we have an eigenbasis, \( (3, 13) \)
then we can group them to bases of
\[ E_{\lambda_1}, \ldots, E_{\lambda_k}. \]
Uniqueness of
\[ V = a_1 v_1 + \ldots + a_n v_n \quad (v_1, \ldots, v_n) \text{ eigenbasis} \]
\[ = \] uniqueness of
\[ V = u_1 + \ldots + u_k, \quad u_j \in E_{\lambda_j}. \]
Had observed before: If \( V = u_1 + \ldots + u_k, \quad u_j \in E_{\lambda_j} \)
then \( u_j \) are uniquely det. by \( V \)
I.e. linear independence of eigenvectors with different eigenvalues.
Therefore given any \( V, T: V \to V, \)
\( E_{\lambda_1}, \ldots, E_{\lambda_k} \) are eigenspaces (distinct)
If \( U = E_{\lambda_1} + \ldots + E_{\lambda_k} = \sum u_1 + \ldots + u_k, \quad u_j \in E_{\lambda_j} \)
\[ = \text{Span} \left( E_{\lambda_1}, u_1, \ldots, u_k \right) \]
\[ \therefore \quad U = E_{\lambda_1} \oplus \ldots \oplus E_{\lambda_k} \]
\[ \therefore \quad \text{I admit an eigenbasis} \quad \Rightarrow \quad V = U \quad \Rightarrow \quad \text{all vectors are sums of eigenvectors.} \]
Think of direct sums \( V_1 \oplus \ldots \oplus V_k \) as Sketch:
"\( V_1, \ldots, V_k \) are linearly ind.,
\( T \) they span \( V_1 \oplus \ldots \oplus V_k. \)"
Exercise (or HW?) $V_1 + \ldots + V_k$ is direct $(4/13)$

$v_1, \ldots, v_k \subseteq V = \text{vs.}$

(i.e. $\forall v \in V_1 + \ldots + V_k \exists ! v_j \in V_j, v_1 + \ldots + v_k = v$)

$
\iff \dim (V_1 + \ldots + V_k) = \dim V_1 + \ldots + \dim V_k
$

Go back to thm: $F^n = V \oplus V^\perp \forall V$.

Saying:

- $F^n = V + V^\perp$ (more statement)
- $\dim V + \dim V^\perp = n$
- $V \cap V^\perp = \{0\}$ (easy part),
  no vector is perpendicular to itself (except $0$).

When $V$ admits an $n$ basis, we proved it by:

$\text{proj}_V : F^n \rightarrow V \text{ defined by}

\text{proj}_V (u) = (\langle u, v_1 \rangle v_1 + \ldots + \langle u, v_k \rangle v_k)

\text{ ( } \langle u, v \rangle := u \cdot \overline{v} \text{ ).}

\text{if } F = \mathbb{R} : u \cdot v.$

Observe: $\text{proj}_V (v) = v \forall v \in V$.
Also observed: \( \ker (\text{proj}_V) = V^\perp. \)

Because \( \ker = \{ w \mid \langle w, v_i \rangle = 0 \ \forall i \} = \{ w \mid \langle w, a_1 v_1 + \ldots + a_n v_n \rangle = 0 \ \forall a_1, \ldots, a_n \in \mathbb{R} \} = \{ w \mid \langle w, V \rangle = 0 \ \forall V = V^\perp \}. \)

**Cor**

If \( u \in \mathbb{R}^n \), we have

\[
    u = \text{proj}_V(u) + (u - \text{proj}_V(u))
    \]

\( V \cap \ker (\text{proj}_V) = \{ 0 \} \)

\[
    \langle u, v \rangle = \langle \text{proj}_V(u), v \rangle \quad \forall v \in V.
    \]

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How to prove \( \exists \) orthonormal basis \( \{ v_i \} \) of \( V \)?

We will use these projections.

**Theorem (Gram-Schmidt)** Given any \( v_1 \in \mathbb{R}^n \)

\( \exists \) orthonormal basis. Moreover, if \( (v_1, \ldots, v_k) \) basis of \( V \), then there exists \( (v_1', \ldots, v_k') \) orthonormal basis s.t. \( v_i' \in \text{Span}(v_1, \ldots, v_i) \).

(e.g. \( v_1' \) is a multiple of \( v_i \) : \( v_i' = \frac{v_i}{\|v_i\|} \)).
Formula: \[ V_j' = V_j - \text{proj}_{\text{span}(v_1, \ldots, v_{j-1})} V_j \] 

This formula makes sense if \( v_1', \ldots, v_{j-1}' \) already
constructed, s.t. \( (v_1', \ldots, v_{j-1}') \) orthonormal,
Since \( \text{span}(v_1, \ldots, v_{j-1}) = \text{span}(v_1', \ldots, v_{j-1}') \) is orthonormal,
\[ \Rightarrow \text{proj}_{\text{span}(v_1, \ldots, v_{j-1})} = \text{proj}_{\text{span}(v_1', \ldots, v_{j-1}')} \exists \text{ by previous thm.} \]
\[ (\text{if } \mathbb{F}^n = W \oplus W^\perp \text{ has orthonormal basis.}) \]

What do we have to prove? \( (v_1, \ldots, v_k) \) basis of \( V \).
Inductively if \( (v_1, \ldots, v_{j-1}) \) constructed and
Orthornormal assume: \( \text{span}(v_1', \ldots, v_k') = \text{span}(v_1, \ldots, v_k) \) \( \forall k \in \mathbb{R} \)
In particular: \( v_k' \in \text{span}(v_1, \ldots, v_k) \) \( \forall k \in \mathbb{R} \)
Need: \( (v_1', \ldots, v_k') \) orthonormal, spans
\( \text{span}(v_1, \ldots, v_k) \)
\[ V_j' = V_j - \text{proj}_{\text{span}(v_1, \ldots, v_{j-1})} V_j = \frac{V_j''}{\|V_j''\|}, V_j'' = \text{num} \]
If nonzero $v_i$, automatically unit $v_i$.

$$\forall \ell < j,$$

But $\langle v_i', v_\ell' \rangle = \frac{\langle v_i'', v_\ell' \rangle}{\|v_i''\|}$.

$$\langle v_i'', v_\ell' \rangle = \langle v_i - \text{Proj}_{\text{span}(v_i, \ldots, v_{j-1})} v_i, v_\ell' \rangle$$

$$= \langle v_i, v_\ell' \rangle - \langle \text{Proj}_{\text{span}(v_i, \ldots, v_{j-1})} v_i, v_\ell' \rangle$$

$$= \langle v_i, v_\ell' \rangle - \langle \text{Proj}_{\text{span}(v_i', \ldots, v_{j-1})} v_i, v_\ell' \rangle$$

$$= 0 \text{ since } \langle u, v \rangle = \langle \text{Proj}_v u, v \rangle \forall v \in V$$

Set $u = v_i$, $V' = \text{span}(v_i', \ldots, v_{j-1}')$, $V'' = \text{span}(v_i, \ldots, v_{j-1})$.

$$v = v_\ell' \text{ above.}$$

\[ \vdash (v_i', \ldots, v_j') \text{ orthogonal} \]

Since $v_i, \ldots, v_j$ linear independent implies $v_i', \ldots, v_j'$ linear independent $v_i'$ nonzero, span of $v_i, \ldots, v_j = \text{span}(v_i', \ldots, v_j')$.  

\[ \square \]
What happened? \( k = 2 \)
\[ \dim V = 2. \]

\( V \) has basis \( (v_1, v_2) \)
\[ v_1' = \frac{v_1}{\|v_1\|} \text{ unit} \]
\[ v_2' = v_2 - \frac{\text{proj}_{\text{Span}(v_1)} v_2}{\|v_2 - \text{proj}_{\text{Span}(v_1)} v_2\|} = \frac{v_2 - \text{proj}_{v_1} v_2}{\|v_2 - \text{proj}_{v_1} v_2\|} \]

General case: project to \( \text{Span}(v_1, \ldots, v_n) \)
\[ = \text{Span}(v_1', \ldots, v_n') \text{ instead} \]
\[ (n-1) = \dim \text{space} \]
Proof works because key observation: \((9/13)\)

\[
\langle u, v \rangle = \left\langle \text{proj}_v u, v \right\rangle \quad \forall v \in V \subseteq \mathbb{F}^n.
\]

\[\text{Line case: Saying: } u \cdot v = \text{proj}_v u \cdot v, \quad \text{proj}_v u = \frac{u \cdot v}{v \cdot v} v.\]

**Examples: Gram-Schmidt in practice:**

**D2:** basis of \(V \) from \([a, 0, 3], [b, c, 0] \) (9/13).

\(a \neq 0, c \neq 0.\)

**G-S:**

\[V_1' = \frac{V_1}{\|V_1\|} = \frac{[a, 0, 3]}{|a|} = \left[ \pm \frac{1}{0}, 0 \right] \text{ same sign as } a.\]

\[V_2'' = V_2 - \text{proj}_{V_1'} V_2 = [b, c] - \frac{[b, c] \cdot [a, 0, 3]}{[a, 0, 3] \cdot [a, 0, 3]} [a, 0, 3] = [b, c] - (\pm b) [0, 1, 0] = [b, c] - b [0, 1, 0].\]

\[V_2^* = \frac{V_2''}{\|V_2''\|} = \left[ 0, \frac{\pm 1}{b} \right] \text{ same sign as } c.\]
\[ V_2 = [b, c] \text{ arbitrary.} \quad (0/13) \]

Concluded: in \( \mathbb{R}^2 \), fixing \( V_1 \), there are only two possibilities for \( V_2 \).

Always true in \( \mathbb{R}^2 \): if \( V_1 = [a, b] \) \( \perp \) \( V_2 = [c, d] \) arb. basis.

\[ V_1' = \frac{[a, b]}{\|V_1\|} = \frac{[a, b]}{\sqrt{a^2 + b^2}} \Rightarrow V_2' \perp V_1' \]

\( \perp \) line \( \perp \) to \( [a, b] \)

\( \perp \) line \( \perp \) to \( [a, b] \)

\( \mathbb{R} [b, -a] \)

Only two unit vectors in here:

\[ \frac{[b, -a]}{\sqrt{a^2 + b^2}} \quad \frac{[-b, a]}{\sqrt{a^2 + b^2}} \]

Generally \( V \subseteq \mathbb{R}^n \) vector space, \( \dim V = k \)

\( V_1', \ldots, V_{k-1}' \) orthonormal set

\( \Rightarrow \exists \) only two vectors \( \perp \) to \( V_1', \ldots, V_{k-1}' \), h.c. \( \text{Span} (V_1', \ldots, V_{k-1}')^\perp \) \( \perp V \) is a line.
\[ \text{Ex: } \mathbb{R}^3 \cong V, \quad \dim V = 2. \]

\[ V = \text{Span}(C_{1,2,0}^1, C_{0,1,-1}^2) \]

\[ V_1 = \frac{C_{1,2,0}^1}{\sqrt{1^2 + 2^2}} = \frac{1}{\sqrt{5}} C_{1,2,0}^1 \]

\[ V_2 = C_{0,1,-1}^2 \]

\[ \text{proj}_{V_1} V_2 = \frac{\langle V_2, V_1 \rangle}{\langle V_1, V_1 \rangle} V_1 = \frac{1}{\sqrt{5}} \left[ 1, 2, 0 \right] \]

\[ \langle V_2, V_1 \rangle = \frac{1}{\sqrt{5}} (1 - 0 + 2 - 1 + 0 - 1) = \frac{2}{\sqrt{5}} \]

\[ V_2'' = V_2 - \frac{2}{\sqrt{5}} V_1 = \left[ 0, 1, -1 \right] - \left( \frac{2}{\sqrt{5}} \right) \left( \frac{1}{\sqrt{5}} C_{1,2,0}^1 \right) = \frac{2}{\sqrt{5}} C_{1,2,0} \]

\[ V_2' = \frac{V_2''}{\| V_2'' \|} = \frac{\left[ -\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, -1 \right]}{\sqrt{\frac{4}{25} + \frac{1}{25} + 1}}. \]

Double check: \( \langle V_2'', V_1' \rangle = 0 \)
Finally, let's do an example where basis has length $\geq 3$:

$v_1 = [1, 0, 0, 0, 0], \; v_2 = [2, 1, 1, 0], \; v_3 = [2, 1, 1, 1, 1]$  

$V_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{11}} [1, 0, 0, 0, 0]$  

$V_2 = v_2 - \frac{V_1}{\|V_1\|} = \frac{1}{\sqrt{11}} [2, 1, 1, 0] - [0, 0, 0, 0] = [0, 1, 1, 0]$  

$V_3 = \frac{v_3}{\|v_3\|} = \frac{1}{\sqrt{2}} [0, 1, 1, 1, 1]$.

$V_3' = V_3 - \frac{V_3}{\|V_3\|} \frac{\text{Span}(V_1, V_2)}{(V_1, V_2) \text{orthonormal}}$.

$\text{Proj}_{\text{Span}(v_1, v_2)} v_3 = \frac{\langle v_3, v_1 \rangle}{\|v_1\|^2} v_1 + \frac{\langle v_3, v_2 \rangle}{\|v_2\|^2} v_2$.

$V_3'' = v_3 - \frac{\langle v_3, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle v_3, v_2 \rangle}{\|v_2\|^2} v_2 = [0, 1, 1, 1, 1] - [0, 1, 1, 0] = [0, 0, 0, 0]$.

$\langle v_3, v_1 \rangle = 2, \quad \|v_1\|^2 = 11, \quad \|v_2\|^2 = 2$.

$\langle v_3', v_1 \rangle = \frac{1}{\sqrt{2}} (0 + 1 + 1 + 0) = \frac{3}{\sqrt{2}}$.

$\langle v_3', v_2 \rangle = \frac{1}{\sqrt{2}} (0 + 2 + 0 + 0) = \frac{3}{\sqrt{2}}$.

$V_3'' = v_3 - (2 \frac{V_1}{\|V_1\|} + \frac{3}{\sqrt{2}} \frac{V_2}{\|V_2\|})$.

$= v_3 - (2 [1, 0, 0, 0] + \frac{3}{\sqrt{2}} \frac{[0, 1, 1, 1, 1]}{\sqrt{2}})$.
\[ v_3' = (2, 0, 0, 0) + \frac{3}{2} (0, 1, 1, 0) \]  \hspace{1cm} (13/13)

\[ \langle v_3', v_3 \rangle = \frac{3}{2} \cdot 1 + \frac{3}{2} \cdot 1 = 3 \]

\[ v_3'' = \begin{bmatrix} 2, 1, 2, 17 \\ -2, \frac{3}{2}, \frac{3}{2}, 0 \end{bmatrix} \]

\[ v_3'' = \begin{bmatrix} 2, 1, 2, 17 \\ -2, \frac{3}{2}, \frac{3}{2}, 0 \end{bmatrix} = (0, -\frac{1}{2}, \frac{1}{2}, 1) \]

Check: \( \langle v_3'', v_1 \rangle = 0 \), \( \langle v_3'', v_2 \rangle = -\frac{1}{2} + \frac{1}{2} = 0 \)

\[ v_3' = \frac{v_3''}{\| v_3'' \|} = \frac{0, -\frac{1}{2}, \frac{1}{2}, 1}{\sqrt{\frac{1}{4} + \frac{1}{4} + 1}} \]

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Rest of class (next lecture): Special Operators.

Next operator: Want an eigenbasis.

Define: A normal operator is one that has an orthonormal eigenbasis.

Example: Shift operator \( T(\vec{a}_0, \vec{a}_1, \vec{a}_2) = (\vec{a}_1, \vec{a}_2, \vec{a}_0) \)

Basis: Fourier basis. || Symmetric matrix of operator \( \langle \vec{v}, T \vec{w} \rangle = \langle \vec{v}, \vec{w} \rangle \)