Midterm exams: Tuesday, Feb 18.

Thursday, March 20.

§1.3 Introduction to proof techniques.

Definition: The converse of a statement "if A then B" is: "if B then A".

I.e. \( A \implies B \) has the converse: \( B \implies A \).

"A implies B"

("if A then B"

Notation: \( \neg A = "\text{not } A" \) (tilde A).

Definition: The inverse of a conditional statement is: \( A \implies B \) has the inverse: \( \neg A \implies \neg B \).

Definition: The contrapositive of \( A \implies B \) is: \( \neg B \implies \neg A \).
Example: \( \text{If it freezes, then campus shuts down.} \)  

Converse: \( \text{If campus shuts down, then it freezes.} \) (false!)

Inverse: \( \text{If it doesn't freeze, then campus doesn't shut down.} \)

Contrapositive: \( \text{If campus doesn't shut down, then it doesn't freeze.} \)

We observe: Converse \( \iff \) Inverse (there are equivalent)

AND Contrapositive \( \iff \) Original Statement.

Note: contrapositive of contrapositive is original statement.

Why: contrapos. is: \( \neg B \Rightarrow \neg A \)

Contrapos. of \( A \Rightarrow B \) = Original Statement.

Converse of converse = Original, inverse of inverse = original.
Example of application to prob.

Take a statement, prove the contrapositive.

\( \iff \) original statement

**Ex:** If \( \| x \| = 0 \), then \( x = 0 \).

**Proof:** \( x \neq 0 \implies \| x \| \neq 0 \) **Contrapositive.**

Proof: Assume \( x \neq 0 \). Since \( x \in \mathbb{R}^n \) we can write:

\[ x = [x_1, \ldots, x_n] \]. \( x \neq 0 \) says: \( x_k \neq 0 \) for some \( 1 \leq k \leq n \).

**Remark:** \( [0, \ldots, 0, 1, 0, \ldots, 0] \neq 0 \),

"standard basis vector"

\( \|x\| = \text{defn of length} \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} \).

\( \|x\|^2 = \text{summing} x_1^2 + x_2^2 + \cdots + x_n^2 \).

**Use:** \( x_1^2, \ldots, x_n^2 \geq 0 \).

\( x_k \neq 0 \implies x_k^2 > 0 \)

Thus \( x_1^2 + \cdots + x_n^2 \geq x_k^2 \)

\[ \begin{bmatrix} x_1^2 & \cdots & x_{k-1}^2 & x_{k+1}^2 & \cdots & x_n^2 \end{bmatrix} \geq 0 \]

\( \implies x_1^2 + \cdots + x_{k-1}^2 + x_{k+1}^2 + \cdots + x_n^2 \geq 0 \)

\( x_k > 0 \)

Conclude: \( \|x\|^2 > 0 \). Thus \( \|x\| \neq 0 \).
Mathematical Induction:

This is (essentially the only) method of proving a statement is true for all (positive, or \( n \geq i \) for some fixed \( i \in \mathbb{Z} \)) integers.

Note: \( A(n) \) is true for all positive integers \( n \)
is the same as:

\( A(n) \) is true, \( n \geq i \), with \( i = 1 \).
[Usually in practice \( i = 0 \) or \( 1 \).]

How it goes:

Show:

\( A(i) \) true

And: \( A(n) \Rightarrow A(n+1) \).

Conclude:

\( A(i) \Rightarrow A(i+1) \Rightarrow \ldots \).

Thus \( A(n) \) true if \( n \geq i \).