The Method of Direct Proof
(Used Only for Proving Universal Statements)

The Design for Direct Proofs of Universal Statements of the form:
\[ \forall x \in D, \text{ predicate } P(x). \]

To Prove: \[ \forall x \in D, \text{ predicate } P(x). \]

Proof: Let \( x \) be any element of the domain \( D. \)
\[ \text{[ N. T. S. P(x) is true about this particular x value.]} \]
\[ \vdots \]
\[ \vdots \]
\[ \therefore P(x). \]
\[ \therefore \text{For every element } x \text{ in } D, \text{ predicate } P(x), \text{ by Direct Proof. QED} \quad \left[ \text{quod erat demonstrandum} \right] \]

For a proof of a universal conditional statement, the proof requires a particular extended form:

The Design for Direct Proofs of Universal IF-THEN Statements:

\[ \forall x \in D, \text{ IF } P(x), \text{ THEN Q(x).} \]

To Prove: \[ \forall x \in D, \text{ IF } P(x), \text{ THEN Q(x).} \]

Proof: Let \( x \) be any element of the domain \( D. \) [ First, define your variables! ]

Suppose \( P(x). \) [ Suppose the "IF" part! ]

\[ \text{[ N. T. S. Q(x) is true about this particular x value.]} \]
\[ \vdots \]
\[ \vdots \]
\[ \therefore Q(x). \quad \text{[ Conclude the "THEN" part, proving "If } P(x), \text{ Then } Q(x)" \text{ about the generic value of } x. ] \]
\[ \therefore \text{For every element } x \text{ in } D, \text{ IF } P(x), \text{ THEN Q(x), by Direct Proof.} \]

\[ \text{QED} \]

Note: The two statements "Let } x \text{ be any ..." and "Suppose } P(x)" can be combined in the single statement:

"Let } x \text{ be any element of the domain } D \text{ such that } P(x)." 

As an example, the beginning on a proof of the statement, "For all integers } n, \text{ if } n \text{ is odd, then } n^2 + 1 \text{ is even}," could begin as follows (and in either form):

"Let } n \text{ be an integer.}
Suppose that } n \text{ is an odd number.}
\[ \text{[ NTS: } n^2 + 1 \text{ is even]} \]

"Let } n \text{ be an integer such that } n \text{ is odd}."
\[ \text{[ NTS: } n^2 + 1 \text{ is even]} \]
The Method of Direct Proof
(Used Only for Proving Universal Statements)

The method of "Direct Proof" is used only for proving a statement which is quantified by a universal quantifier, e.g., "For all . . .", "For every . . .", etc.

Two other proof methods (Mathematical Induction and Proof-by-Contradiction) can also be used for the proof of a universal statement, but the method of Direct Proof can only be used in proving a universal statement.

If the statement-to-prove is not a universal statement,
then DO NOT WRITE "by Direct Proof" at the end of the proof.

Definition of "Direct Proof Method": A proof using the "Direct Proof Method" is one in which a chosen variable represents an arbitrary element of the domain and the predicate of the universal statement is proved to be true about that arbitrary element.

The simplest form of a universal statement (in symbolic terms) is: \( \forall x \in D, \, \text{predicate } P(x) \).

[Note: \( D \) is the domain of the variable \( x \) and \( P(x) \) is a predicate which makes an assertion about the entity that the variable \( x \) represents, e.g., "For every integer \( x \), \( x^2 \geq 0 \)," in which \( D = \mathbb{Z} \) and \( P(x) = " x^2 \geq 0."

The Steps in the Method of Direct Proof

1) Begin with a statement which defines a variable, (say, for example, \( x \)), which represents a particular but arbitrarily chosen element of the domain \( D \).

2) Operating with expressions in terms of \( x \), make a logical argument which concludes essentially,
   \[ \text{"Therefore, } P(x) \text{ is true."} \]
   (This statement is referring only to the particular but arbitrarily chosen value of \( x \), selected at the start.)

3) When the truth of \( P(x) \) has been established regarding the particular but arbitrarily chosen value of \( x \), then one can conclude that the \( P(x) \) is true for all values of \( x \), and one writes:
   \[ \text{"Therefore, for all } x \in D, \text{ predicate } P(x) \text{ is true, by Direct Proof."} \]

[Note: The most common form of the predicate \( P(x) \) is that of a conditional (If-Then) statement, \( A \to B \); thus, often the form of the universal statement to be proved is: \( \forall x \in D, \, P(x) \to Q(x) \).

When this is the case, the definition of the generic particular variable \( x \) is followed by the supposition that the "IF" part of the conditional statement is true, that is, by "Suppose \( P(x) \)."
This should be followed by the comment \[ \text{NTS } Q(x) \], where \( \text{NTS} \) means "need to show." ]

When the conclusion "Therefore, \( Q(x) \)" is achieved, the conditional "If \( P(x), \) Then \( Q(x) \)" has been proved, and a conclusion of the universal statement ends the proof.

[Note: The defining of the particular but arbitrarily chosen element \( x \) from domain \( D \) can be accomplished with one of three wordings:

1) "Let \( x \) be any . . . (such and such)", as in "Let \( x \) be any integer."

2) "Let (such and such) \( x \) be given", as in "Let integer \( x \) be given" or "Let \( x \in \mathbb{Z} \) be given",
   (writing "Let \( x \in \mathbb{Z} \)", without the "be given", is incorrect.) ]
A Good Proof and a Bad Proof

A Good Proof:

To Prove: The sum of an even integer and an odd integer is odd.

[ The form in symbols: \( \forall m \in \mathbb{Z}, \forall n \in \mathbb{Z}, \) if \( m \) is even and \( n \) is odd, then \( m + n \) is odd. ]

Proof: Let \( m \) and \( n \) be any integers.

Suppose \( m \) is even and \( n \) is odd. \hspace{1cm} \text{[ NTS: } m + n \text{ is odd. ]}

\( \therefore \) By the definitions of "even" and "odd", there exist integers \( s \) and \( t \) such that
\[
m = 2s \quad \text{and} \quad n = 2t + 1.
\]
\( \therefore m + n = 2s + (2t + 1), \) by substitution,
\( = 2(s + t) + 1, \) by Rules of Algebra.

Let \( k = s + t \), which is an integer since sums and products of integers are integers.

So, \( m + n = 2k + 1, \) by substitution.

\( \therefore m + n \) is odd, by definition of "odd".

\( \therefore \) The sum of an even integer and an odd integer is an odd integer, by Direct Proof.  \hspace{1cm} \text{QED}

Good Points about the Good Proof:

"Let" has been used for the initial definition of the variables (rather than "Suppose").

"Suppose" has been used for further restricting the domains of the variables.

Every deduction is justified with a reason.

Every comment is enclosed in brackets:  [ COMMENT ]

[ A "Comment" is a statement which does not advance the proof but which only aids the reader (and often the writer too) in understanding what is going on in the proof. ]

The statement-to-prove is re-formulated as an equivalent "IF-Then" statement at the start [and in a comment] to aid the reader (and the proof writer, too).

The calculation \( s + t \),
which is the integer required by the definition of "odd" to prove that \( m + n \) is odd,
is used to define a third variable, \( k \), so that the proof includes the statement,

\( \therefore m + n = 2k + 1, \) rather than the statement,

\( \therefore m + n = 2(\text{some integer}) + 1, \) which is NOT ALLOWED, but which you will see in the book.
A Bad Proof:

To Prove: The sum of an even integer and an odd integer is odd.

Proof: Suppose $2s$ and $2t + 1$ are arbitrarily chosen integers.

$$ (2s) + (2t + 1) = 2(s + t) + 1 $$

$$ \Rightarrow $$

$$ (2s) + (2t + 1) = 2(\text{some integer}) + 1. $$

$\therefore$ The sum of an even integer and an odd integer is an odd integer, by Direct Proof. QED

Bad Points about the Bad Proof:

$s$ and $t$ are not adequately defined as representing integers.

You must separately define $m$ and $n$ as an even integer and odd integer, respectively, in one step, and then, in another step, apply the definitions of "even" and "odd" to define integers $s$ and $t$.

The "some integer" factor is not allowed as discussed above.

The use of logical symbols, such as $\exists$, $\forall$, $\sim$, $\rightarrow$, $\Leftrightarrow$, $\Rightarrow$, are not allowed in proof statements

(but they may be used in COMMENTS, which are placed in brackets [see directly below]).

The use of symbols, including logical symbols, such as $\exists$, $\forall$, $\sim$, $\rightarrow$, $\Leftrightarrow$, $\Rightarrow$, are allowed in proofs written in words when they are used in COMMENTS. [COMMENTS must always be placed in brackets!]

Also, do not use the word "If" when the meaning is "Because". Use "Because" or "Since".

Ex: INCORRECT! $\rightarrow$ "Let $n$ be an even number. If $n$ is even, then $n = 2k$ for some integer $k$."

CORRECT! $\rightarrow$ "Let $n$ be an even number. Because $n$ is even, $n = 2k$ for some integer $k".$$