Example Proofs Involving Divisibility  

(NTS = "Need to Show")

To Prove: For all integers \( m \) and \( n \), if \( 6 \mid m \) and \( 4 \mid n \), then \( 2 \mid (5m - 7n) \).

Proof: Let \( m \) and \( n \) be integers.
Suppose that \( 6 \mid m \) and \( 4 \mid n \).
Then, \( m = 6k \) and \( n = 4p \) for some integers \( k \) and \( p \), by definition of “divides”.
\[ 5m - 7n = 5 (6k) - 7 (4p), \]  
by substitution,
\[ = 30k - 28p \]  
by R. O. A.

[ Need to show: \( 5m - 7n = 2t \) for some integer \( t \).]

From above, \( 5m - 7n = 30k - 28p \)
\[ = 2 (15k - 14p) \]  
by R. O. A.

Let \( t = 15k - 14p \), which is an integer.
\[ \therefore 5m - 7n = 2t \]  
by substitution, and \( t \) is an integer.
\[ \therefore 2 \mid (5m - 7n) \]  
by definition of “divides”.
\[ \therefore \text{For all integers } m \text{ and } n, \text{ if } 6 \mid m \text{ and } 4 \mid n, \text{ then } 2 \mid (5m - 7n), \text{ by Direct Proof.} \]

Q E D

Theorem 4.3.3 (Page 137): “Divisibility is Transitive;” that is, for all integers \( a \), \( b \), and \( c \), if \( a \mid b \) and \( b \mid c \), then \( a \mid c \).

Proof: Let \( a \), \( b \), and \( c \) be any integers.
Suppose \( a \mid b \) and \( b \mid c \). [NTS: \( a \mid c \). NTS \( c = at \) for some integer \( t \).]
By definition of “divides,” there exist integers \( k \) and \( p \) such that \( b = ak \) and \( c = bp \).
\[ \therefore c = (ak)p \] by substitution of \( b \) by \( (ak) \) in the equation \( c = bp \),
\[ = a(kp) \]  
by R. O. A.
Let \( t = kp \), which is an integer, because products of integers are integers.
\[ \therefore c = at \]  
by substitution, and \( t \) is an integer.
\[ \therefore a \mid c \), by definition of "divides".
\[ \therefore \text{For all integers } a \text{, } b \text{, and } c \text{, if } a \mid b \text{ and } b \mid c \text{, then } a \mid c \text{, by Direct Proof.} \]

Q E D
**To Prove:** For all integers a, b, and c, if \( a \mid b \) and \( a \mid c \), then \( a \mid (b + c) \).

**Proof:** Let \( a, b, \) and \( c \) be any integers.
Suppose that \( a \mid b \) and \( a \mid c \).

Then, \( b = ak \) and \( c = ap \) for some integers \( k \) and \( p \) by definition of “divides”.

\[
\therefore (b + c) = ak + ap \quad \text{by substitution,}
\]
\[
= a(k + p) \quad \text{by R. O. A.}
\]
\[
= at, \quad \text{where} \ t \text{ is the integer such that} \ t = (k + p).
\]

\[
\therefore (b + c) = at, \quad \text{and} \ t \text{ is an integer.}
\]

\[
\therefore a \mid (b + c) \quad \text{by definition of “divides.”}
\]

\[
\therefore \text{For all integers} \ a, b, \text{and} \ c, \text{if} \ a \mid b \text{ and} \ a \mid c, \text{then} \ a \mid (b + c), \text{by Direct Proof.}
\]

\(QED\)